DS-GA.3001 Embodied Learning and Vision

Mengye Ren

NYU

Spring 2025

embodied-learning-vision-course.github.io

Lecture Slides for Note Taking

Logistics

- Next week will be the last lecture.
- Student topic presentation begins.

	Component 1	Component 2
Week 6 (Feb 27)	End-to-End Planning, Continual Learning (1.5 hr)	Tutorial: LLM Agents
Week 7 (Mar 6)	Continual Learning, Few-shot Learning (1 hr)	**Deep Learning for Structured Prediction** Kangrui Yu, Tanishq Sardana, Qing Mu, Owais Shuja
Week 8 (Mar 13)	Guest Lecture – Prof. Wei-Chiu Ma (1 hr)	**3D Vision and Mapping** Sihang Li, Kanishkha Jaisankar, Denis Mbey Akola, Zijin Hu
Week 9 (Mar 20)	**SSL and Object Discovery** Anurup Naskar, Dahye Kim, Sal Yeung, Surbhi (1.5 hr)	**World Model 1** Sidhartha Reddy Potu, Andrew Deur
Week 10	Spring Break.	

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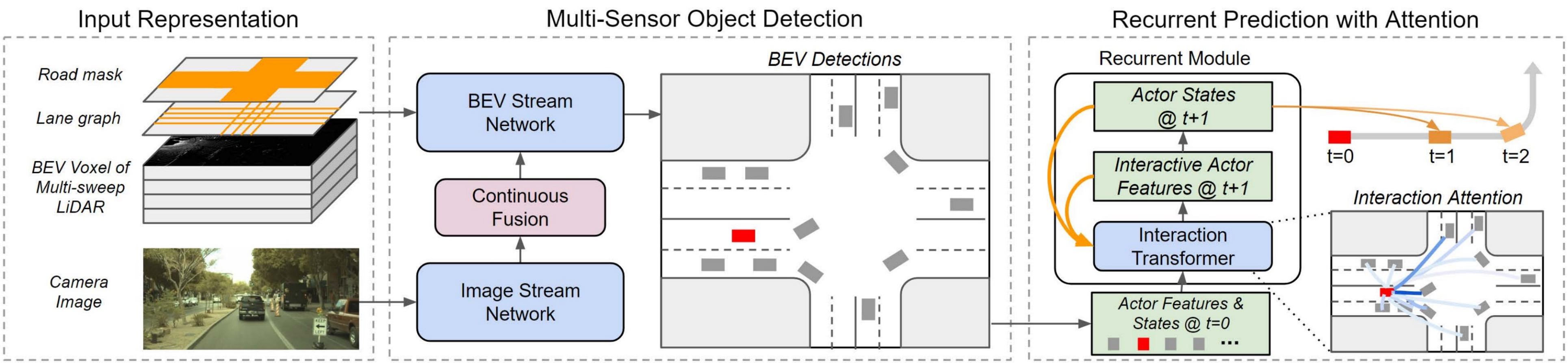
Multi-Agents Joint Prediction

- Joint predict future trajectories by attending to other actors.



Li et al. End-to-end Contextual Perception and Prediction with Interaction Transformer. IROS 2020.

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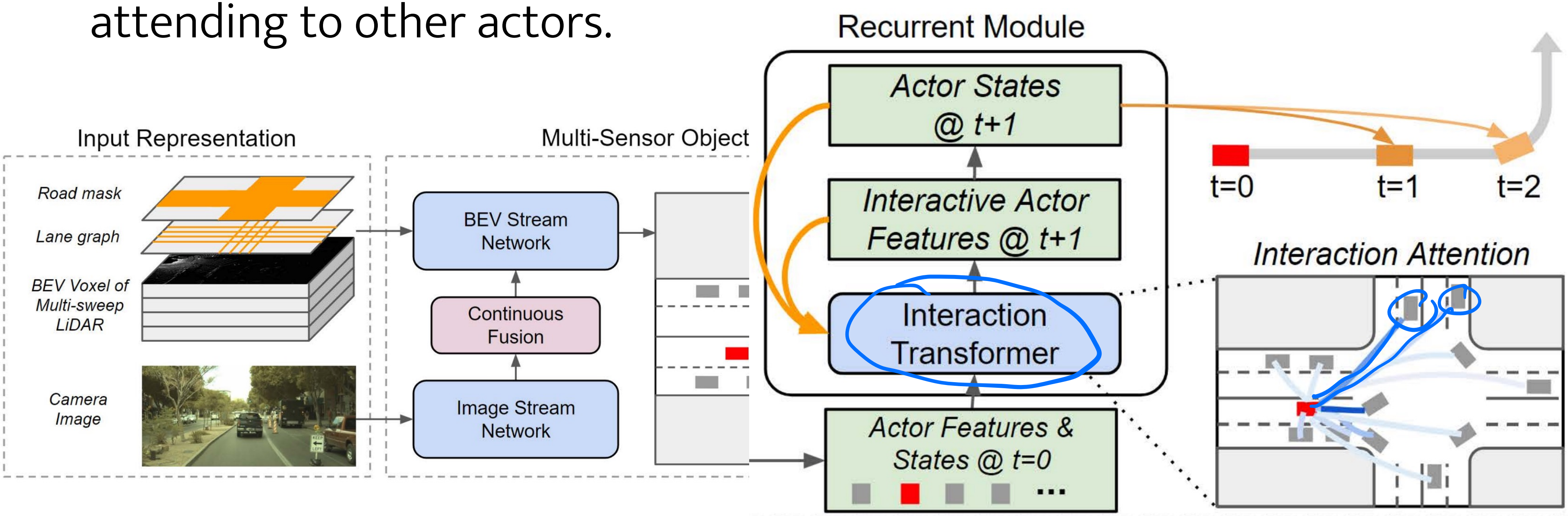
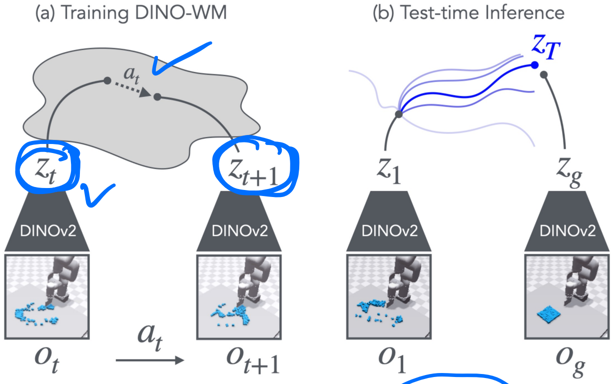
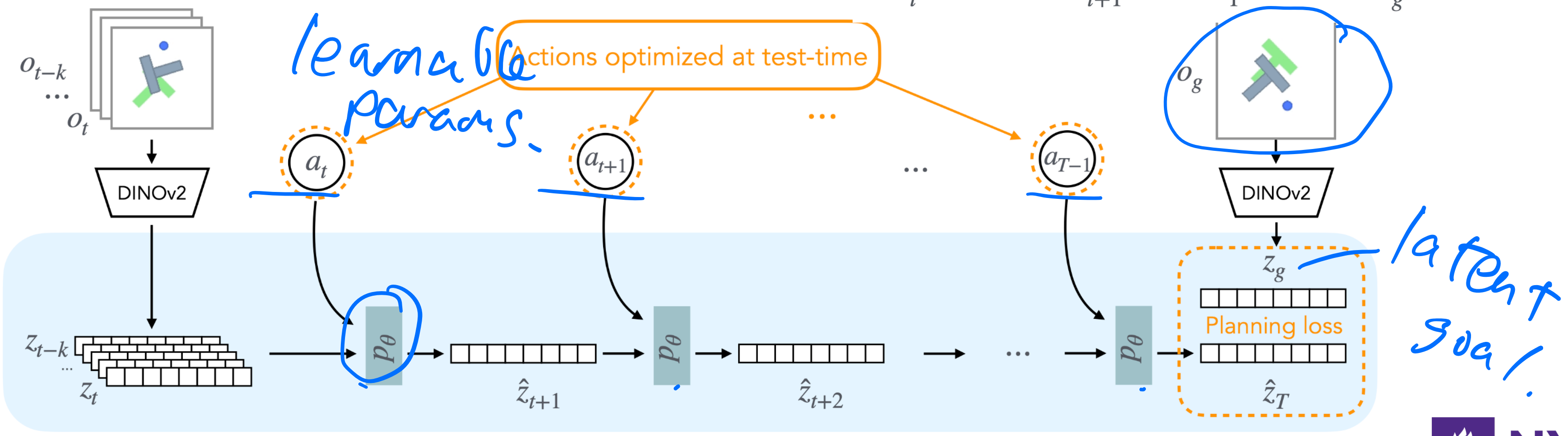


Li et al. End-to-end Contextual Perception and Prediction with Interaction Transformer. IROS 2020.

Latent Prediction + MPC

- Using MPC on the latent space of pretrained visual encoders.
- Learn a predictor of latent states conditioned on actions.





Zhou et al. *DINO-WM: World Models on Pre-trained Visual Features enable Zero-shot Planning.* 2024.

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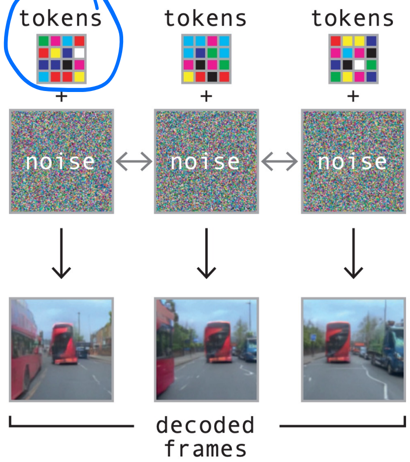
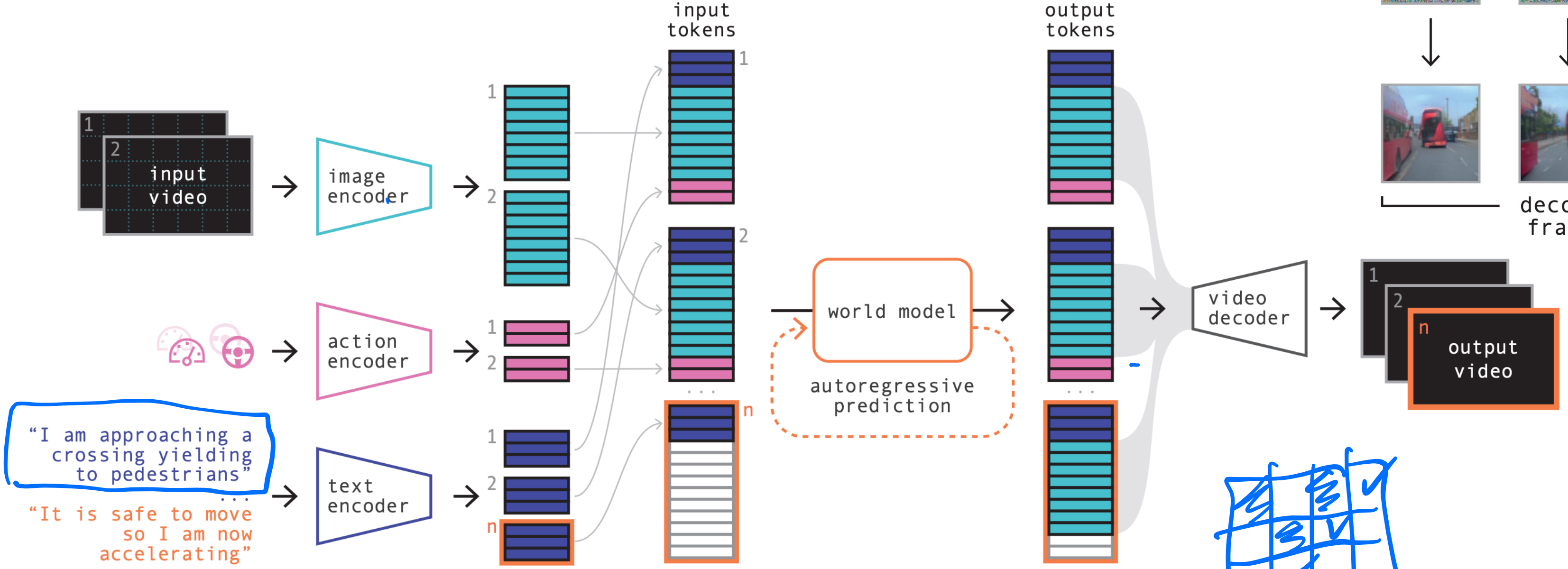
World Model in Video Prediction

Mask GIT.

- Text+action conditioned generation. Diffusion decoder.





Hu et al. GAIA-1: A Generative World Model for Autonomous Driving. arXiv 2023.

WAYVE


4X SPEED
Generated by GAIA-1
4X SPEED
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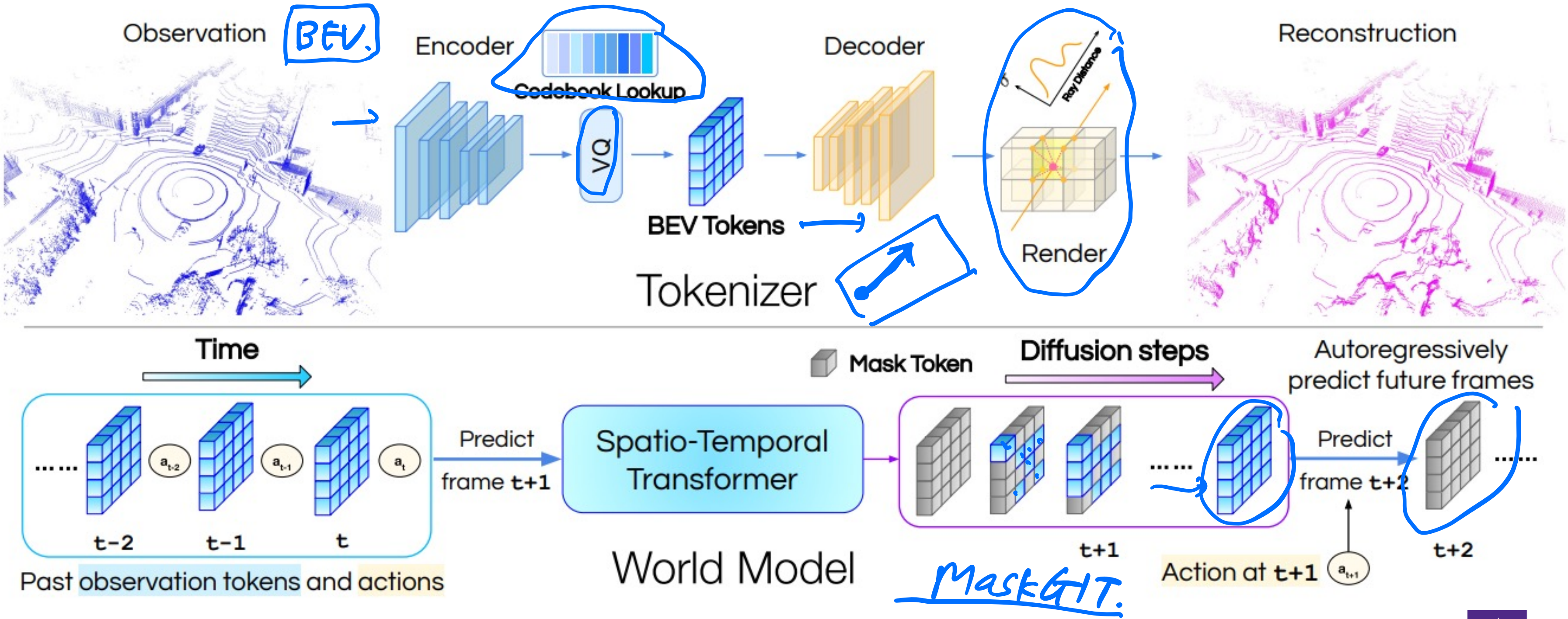
World Model in 3D volume prediction

- Autoregressively predict future 3D point clouds.



Zhang et al. Copilot4D: Learning Unsupervised World Models for Autonomous Driving via Discrete Diffusion. ICLR 2024.

Summary: World Models

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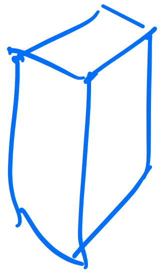

- Explicit object representation
  - Traditional, light weight, instance-specific, hard to learn jointly

Summary: World Models

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  - Traditional, light weight, instance-specific, hard to learn jointly
- Differentiable occupancy, motion field
  - Relatively heavy, spatially grounded, end-to-end learnable

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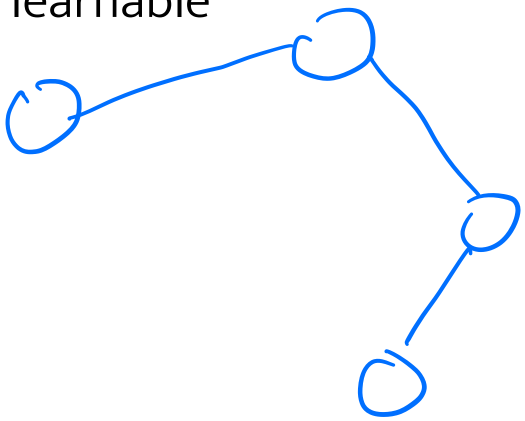
  - Relatively heavy, spatially grounded, end-to-end learnable
- Global latent, RNNs, graph landmarks
  - General-purpose, unstructured



Summary: World Models

- Explicit object representation
  - Traditional, light weight, instance-specific, hard to learn jointly
- Differentiable occupancy, motion field
  - Relatively heavy, spatially grounded, end-to-end learnable
- Global latent, RNNs, graph landmarks
  - General-purpose, unstructured
- Raw video/3D prediction
  - Expensive, good for simulation

Imitation Learning

- The explicit policy model, supervised learning (behavior cloning)

$$\hat{a} = f_\theta(x) \qquad \mathcal{L} = \min_i \|a_i - \hat{a}\|_2^2 \qquad \mathcal{L} = -\log \hat{a}_j$$

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Imitation Learning

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- Energy-based (cost-based) approach

$$\tau^\star = \operatorname{argmin}_\tau E(x, \tau)$$

$$p(\tau \mid x) = \frac{\exp(E(x,\tau))}{\int_\tau \exp(E(x,\tau))}$$

Imitation Learning

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- Energy-based (cost-based) approach

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- Dataset Aggregation (DAgger)
  - Learned policy may deviate from experts
  - Need to collect more groundtruths

Initialize $\mathcal{D} \leftarrow \emptyset$.
Initialize $\hat{\pi}_1$ to any policy in Π.
for $i = 1$ **to** N **do**
  Let $\pi_i = \beta_i \pi^* + (1 - \beta_i)\hat{\pi}_i$.
  Sample T-step trajectories using π_i.
  Get dataset $\mathcal{D}_i = \{(s, \pi^*(s))\}$ of visited states by π_i and actions given by expert.
  Aggregate datasets: $\mathcal{D} \leftarrow \mathcal{D} \bigcup \mathcal{D}_i$.
  Train classifier $\hat{\pi}_{i+1}$ on $\mathcal{D}$.
end for
Return best $\hat{\pi}_i$ on validation.

Algorithm 3.1: DAGGER Algorithm.

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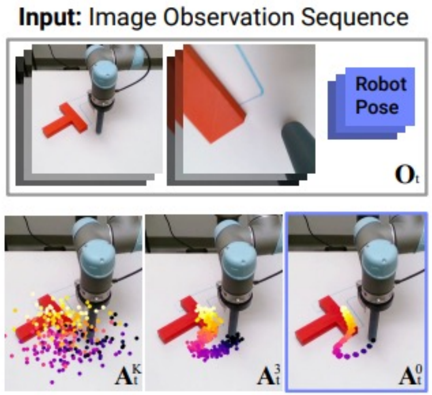
Direct Policy Learning from Diffusion

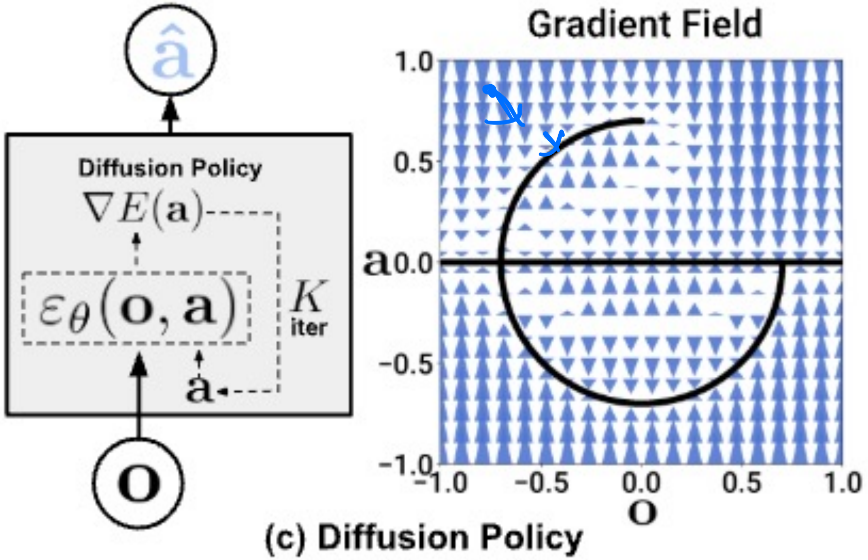
- Error prediction network is conditioned on observation features.

$$A_t^{k-1} = \alpha(A_t^k - \gamma\epsilon_\theta(O_t, A_t^k, k) + \mathcal{N}(0, \sigma^2 I)).$$

$$\mathcal{L} = MSE(\epsilon^k, \epsilon_\theta(O_t, A_t + \epsilon^k, k)).$$

A_t^0





Chi et al. Diffusion Policy: Visuomotor Policy Learning via Action Diffusion. RSS 2023

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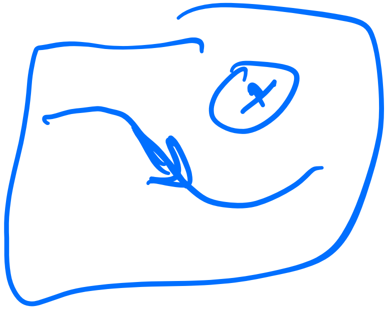
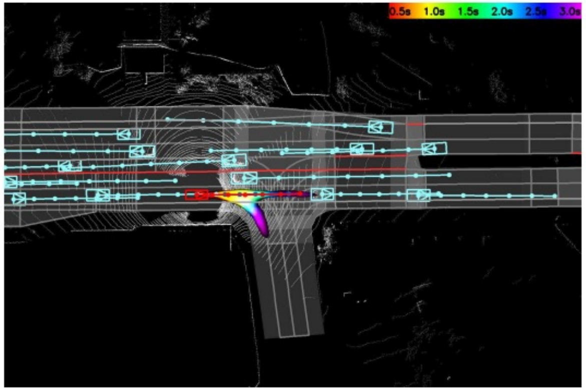
Cost/Value Volume Reasoning

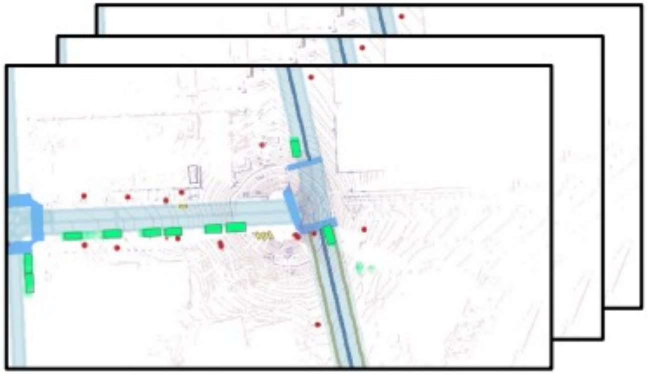
- Interpretability (both costs and planner inputs)



Rasterization



Semantic Occupancy



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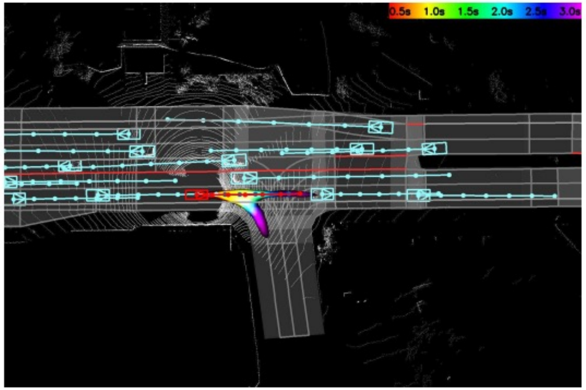
Cost/Value Volume Reasoning

$*\alpha$

obstacles.

- Interpretability (both costs and planner inputs)
- Use spatial geometry to form cost from explicit objects

Rasterization



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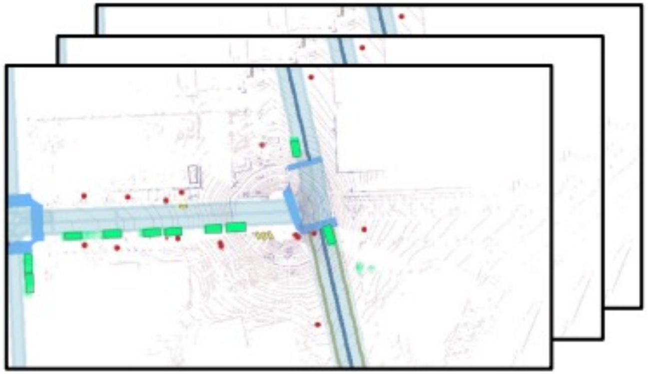
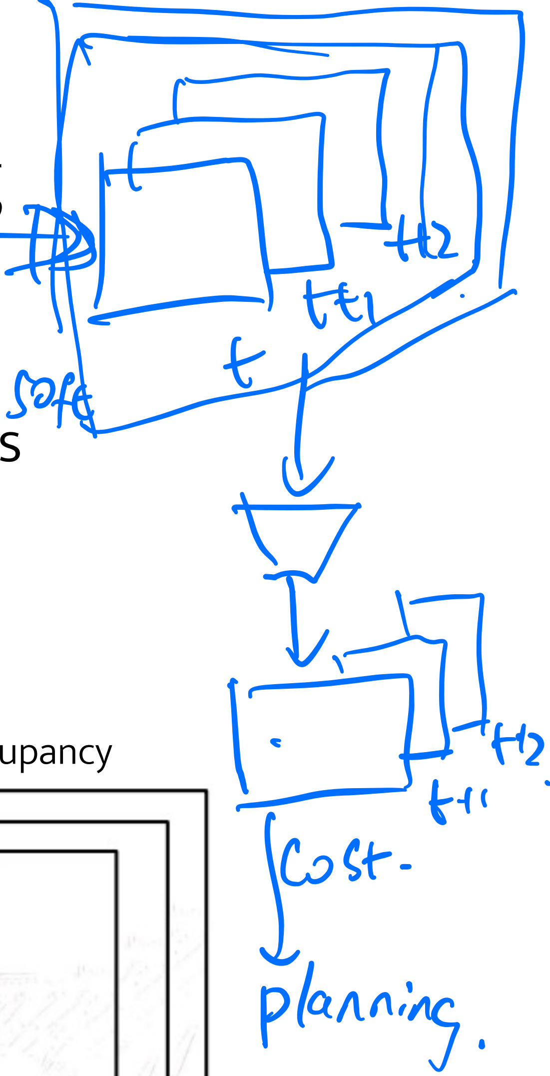


Cost/Value Volume Reasoning

- Interpretability (both costs and planner inputs)
- Use spatial geometry to form cost from explicit objects
- Predict spatial cost volume
  - Rasterize the scene for spatial inputs
  - Predict soft occupancy volumes (present and future)



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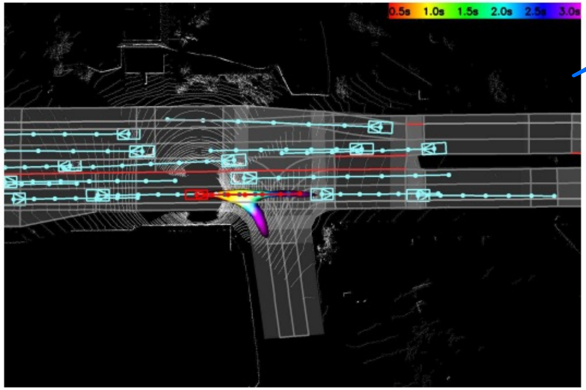

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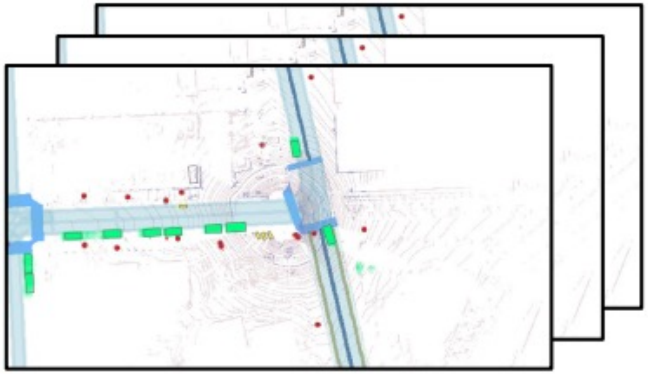

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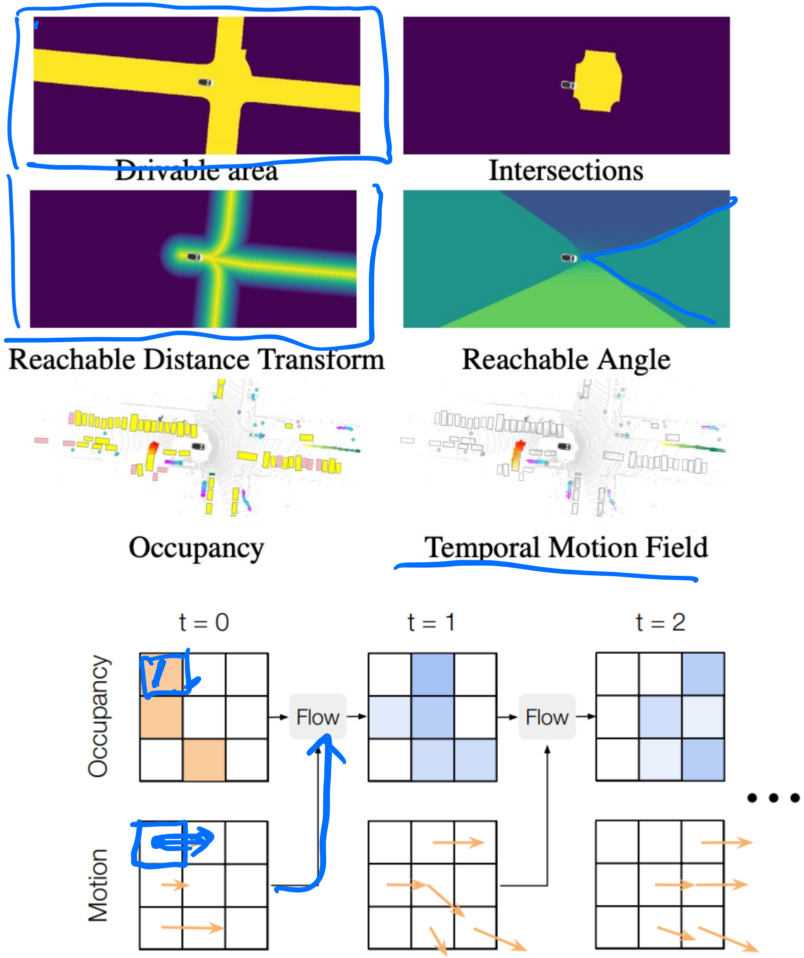
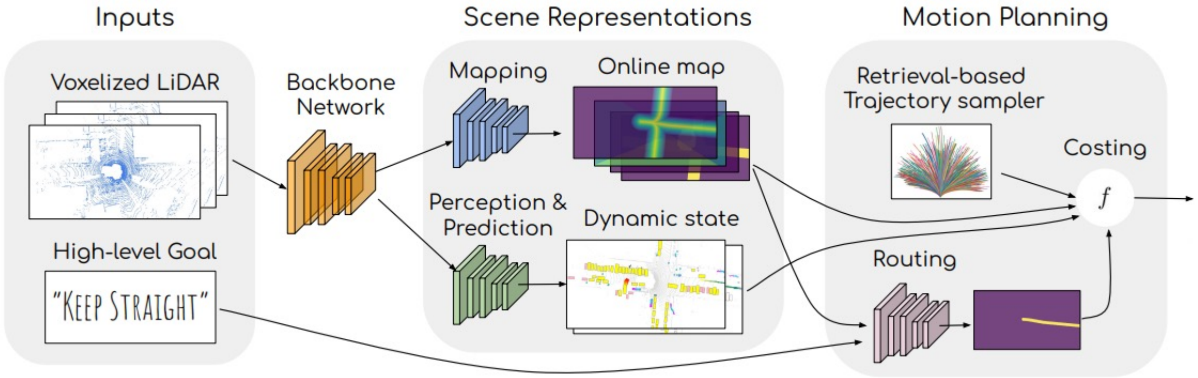
Learning Through Interpretable Predictions

- Semantic occupancy, motion field, mapping, etc. as intermediate predictions.

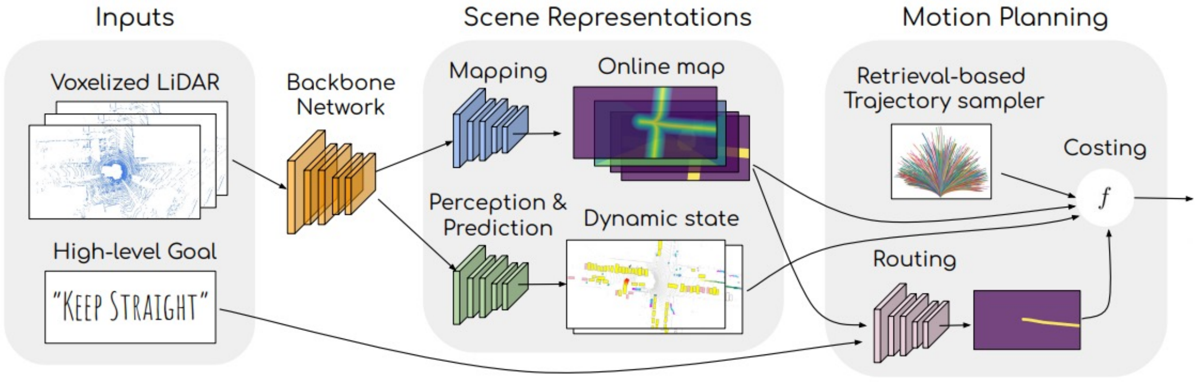


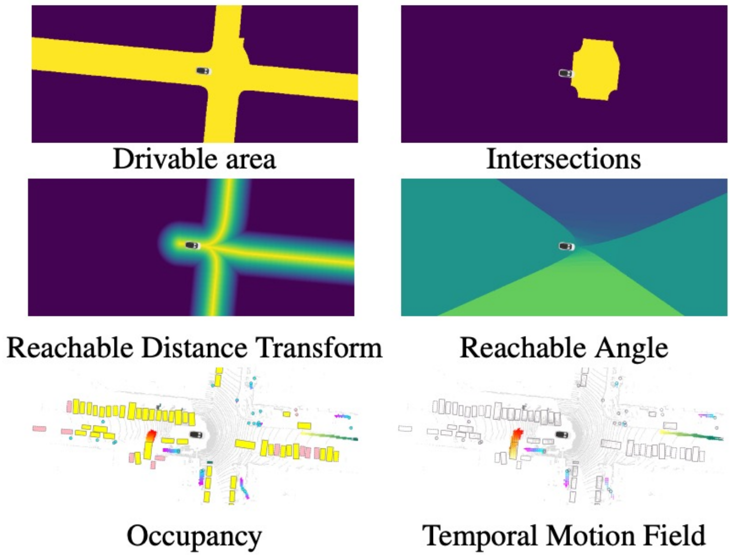


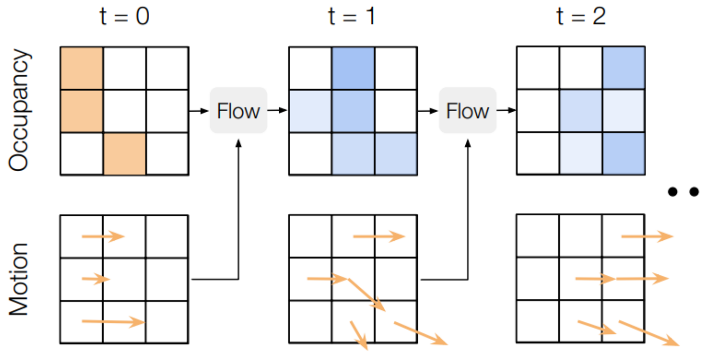
Casas et al. MP3: A Unified Model to Map, Perceive, Predict and Plan. CVPR 2021.

Learning Through Interpretable Predictions

- Semantic occupancy, motion field, mapping, etc. as intermediate predictions.
- Differentiable, supports end-to-end interpretable learning from perception to planning.







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Max-Margin Planning with Explicit Cost Volume

- If we have an explicit cost volume, the cost of a trajectory can be directly queried.

Zeng et al. End-to-end Interpretable Neural Motion Planner. CVPR 2019.

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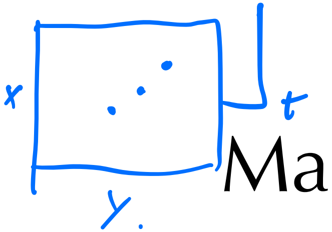

Max-Margin Planning with Explicit Cost Volume

- If we have an explicit cost volume, the cost of a trajectory can be directly queried.
- We can use the max-margin objective to make the groundtruth trajectory have lower costs.

$$\operatorname*{argmin}_{\theta} \sum_{\{(\hat{x}_i^t, \hat{y}_i^t)\}_{i=1\ldots N}} \max_{i} \sum_{t=1}^{T} \underbrace{C_\theta^t[x_t, y_t]}_{\text{expert}} \underbrace{- C_\theta^t[\hat{x}_i^t, \hat{y}_i^t]}_{\text{model.}} + d_i^t$$

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Max-Margin Planning with Explicit Cost Volume

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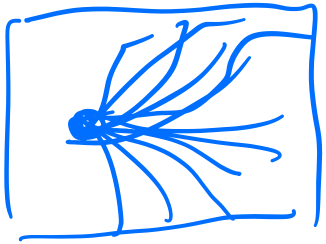
- Find the lowest cost trajectory among a batch of samples.



✓ dynamics

$$\operatorname*{argmin}_{\theta} \sum_{\{(\hat{x}_i^t, \hat{y}_i^t)\}_{i=1\ldots N}} \max_{i} \sum_{t=1}^{T} C_\theta^t[x_t, y_t] - C_\theta^t[\hat{x}_i^t, \hat{y}_i^t] + d_i^t$$

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Max-Margin Planning with Explicit Cost Volume

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- Low-dimensional/known dynamics problems: External samplers

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- Low-dimensional/known dynamics problems: External samplers
- In general, needs to perform optimization (e.g. DP)

$$\underset{\theta}{\operatorname{argmin}} \sum_{\{(\hat{x}_i^t, \hat{y}_i^t)\}_{i=1 \ldots N}} \max_i \sum_{t=1}^{T} C_\theta^t[x_t, y_t] - C_\theta^t[\hat{x}_i^t, \hat{y}_i^t] + d_i^t$$

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EBM Planning

- Energy-based framework also needs negative samples.

Florence et al. Implicit Behavioral Cloning. CoRL 2021.

EBM Planning

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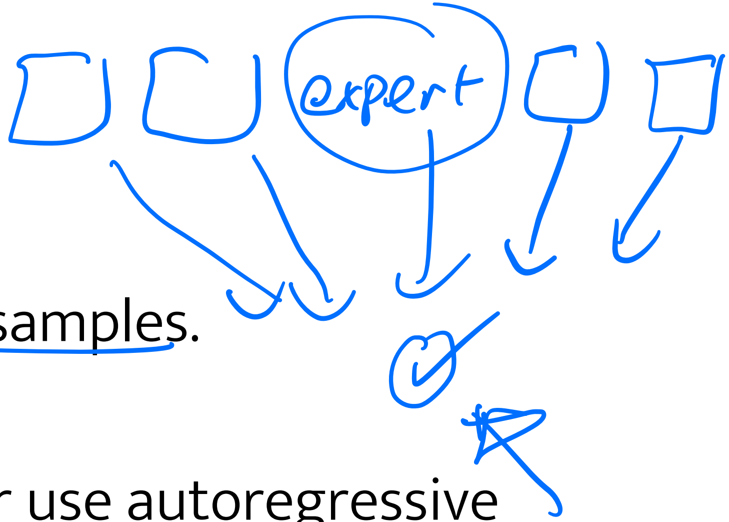
EBM Planning



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- If there isn’t an external sampler, we can either use autoregressive energy of sampling one dimension at a time, or gradient-based Langevin MCMC.

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Langevin MCMC: $$\tilde{\mathbf{y}}_i^k = \tilde{\mathbf{y}}_i^{k-1} - \lambda \left(\frac{1}{2} \nabla_{\mathbf{y}} E_\theta(\mathbf{x}_i, \mathbf{y}_i^{k-1}) + \omega^k \right), \omega^k \sim \mathcal{N}(0, \sigma).$$

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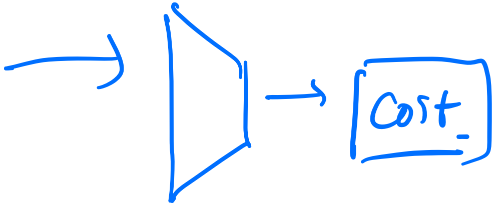
Loss: $$\mathcal{L} = \underline{\textstyle\sum_i - \log(p_\theta(\mathbf{y}_i \mid \mathbf{x}, \{\tilde{\mathbf{y}}_i\}_j)}$$ $$p_\theta(\mathbf{y}_i \mid \mathbf{x}, \{\tilde{\mathbf{y}}_i\}_j = \frac{e^{-E_\theta(\mathbf{x}_i, \mathbf{y}_i)}}{e^{-E_\theta(\mathbf{x}_i, \mathbf{y}_i)} + \sum_j e^{-E_\theta(\mathbf{x}_i + \tilde{\mathbf{y}}_{i,j})}}$$

Florence et al. Implicit Behavioral Cloning. CoRL 2021.

Value Iteration Networks



- A network design for predicting cost volumes that are grounded from the classic value iteration algorithm.

Tamar et al. *Value Iteration Networks*. NeurIPS 2016.

Value Iteration Networks

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Value Iteration Networks

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$$V^*(s) = \max_\pi V^\pi(s)$$

bad policy.

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$$Q_n(s,a) = R(s,a) + \gamma \sum_{s'} P(s'|s,a) V_n(s')$$

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$$Q_n(s,a) = R(s,a) + \gamma \sum_{s'} P(s'|s,a) V_n(s')$$

$$V_{n+1}(s) = \max_a Q_n(s,a)$$

$$\pi^*(s) = \operatorname{argmax}_a Q_\infty(s,a)$$

Tamar et al. Value Iteration Networks. NeurIPS 2016.

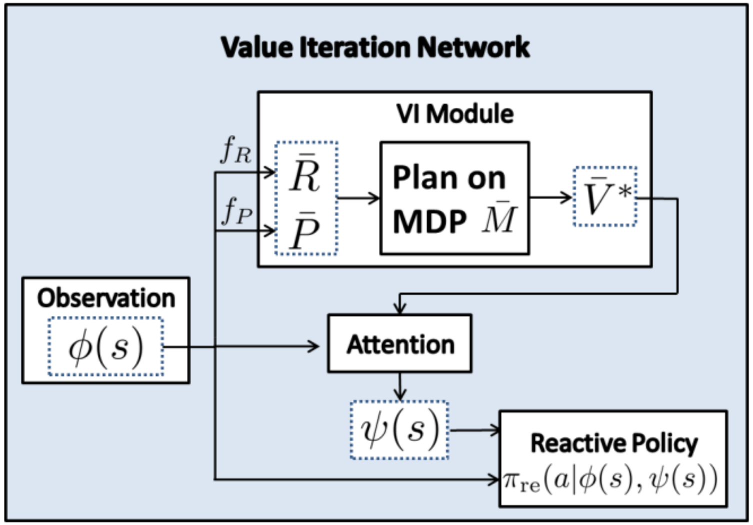
Value Iteration Networks

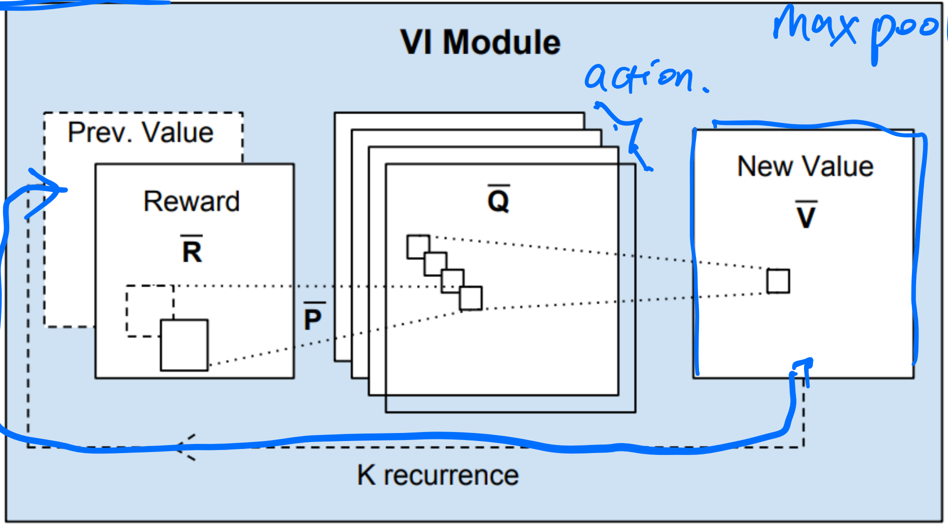
- Reward and previous value are fed into a CNN to generate Q of A channels. Transition matrix is convolutional kernel. Then Max-Pooling.

$$Q_n(s,a) = R(s,a) + \gamma \sum_{s'} P(s'|s,a) V_n(s') \qquad V_{n+1}(s) = \boxed{\max_a Q_n(s,a)}$$

max pooling.

action.





Tamar et al. Value Iteration Networks. NeurIPS 2016.

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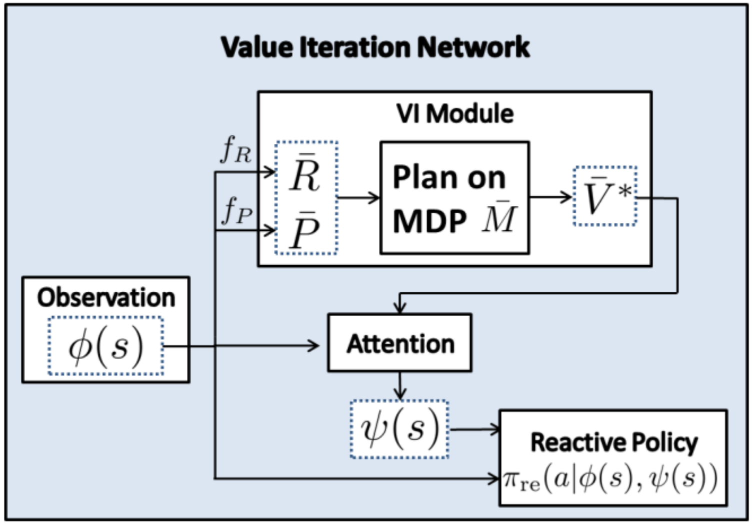
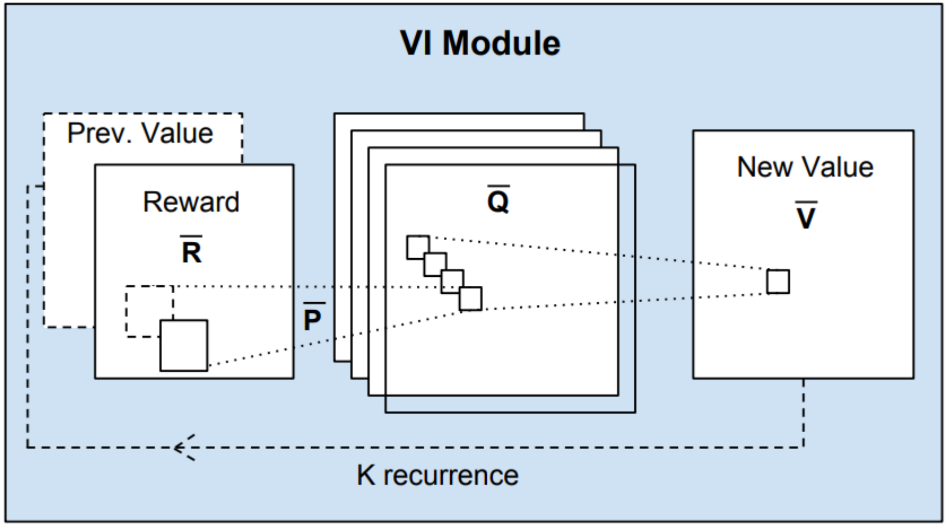
Value Iteration Networks

{ imitation
RL.

- Select the current state and choose an action from softmax.

$$\hat{a} \sim \mathrm{softmax}_a(Q(s,a))$$





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Value Iteration Networks

- A baseline would be to untie the weights through iterations, more like a feedforward CNN.
- Achieve more training data efficiency by imposing the structure.

Training data	VIN			VIN Untied Weights		
	Pred. loss	Succ. rate	Traj. diff.	Pred. loss	Succ. rate	Traj. diff.
20%	0.06	98.2%	0.106	0.09	91.9%	0.094
50%	0.05	99.4%	0.018	0.07	95.2%	0.078
100%	0.05	99.3%	0.089	0.05	95.6%	0.068

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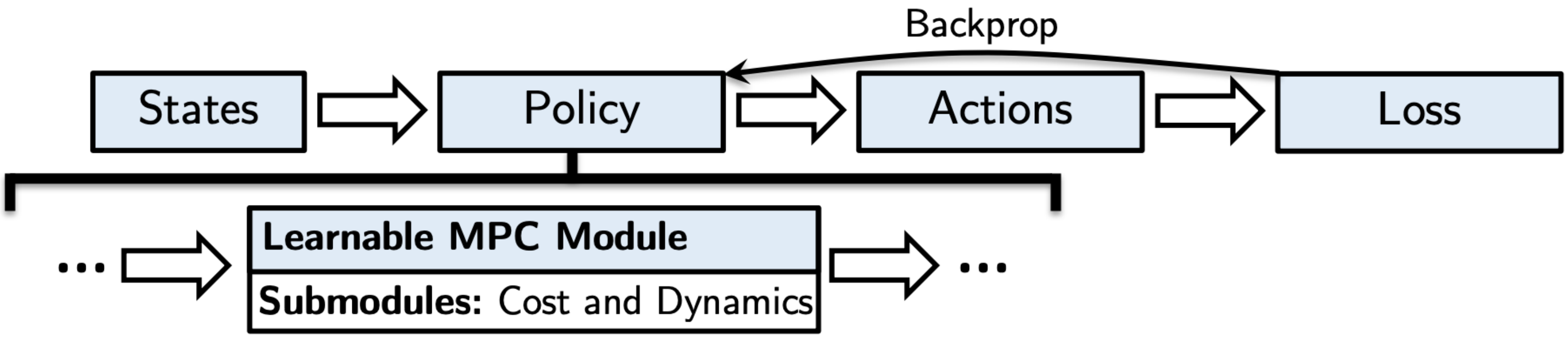
VIN → RNN.

$\underbrace{\text{k steps.}}$

Backprop through Planning

- Treat planning as an end-to-end layer. Can be used for RL/Imitation.



Amos et al. Differentiable MPC for End-to-end Planning and Control. NeurIPS 2018.

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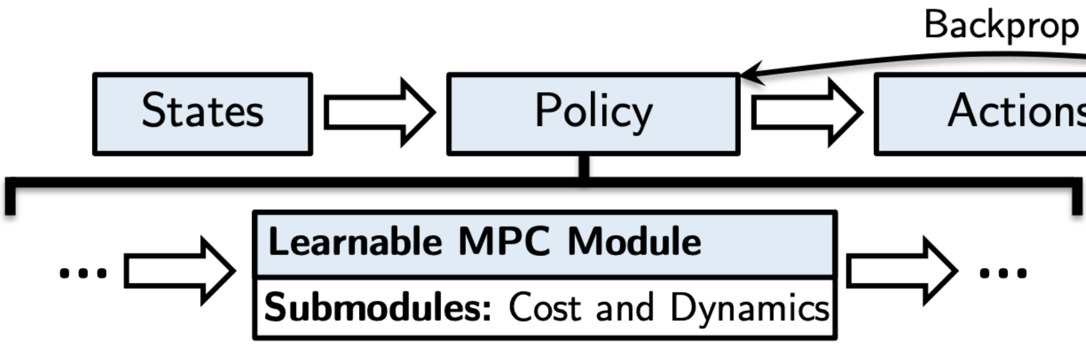
Backprop through Planning

- Treat planning as an end-to-end layer. Can be used for RL/Imitation.
- Option 1: Unrolling a finite number of steps



Amos et al. Differentiable MPC for End-to-end Planning and Control. NeurIPS 2018.

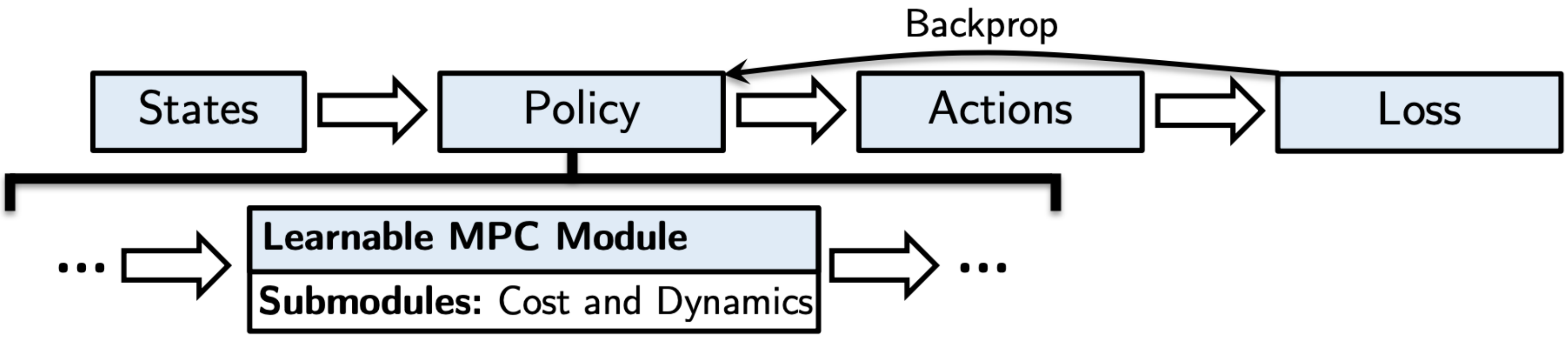
Backprop through Planning

- Treat planning as an end-to-end layer. Can be used for RL/Imitation.
- Option 1: Unrolling a finite number of steps
- Option 2: Solve till convergence, backprop for a finite step

K

a^*.

T-BPTT



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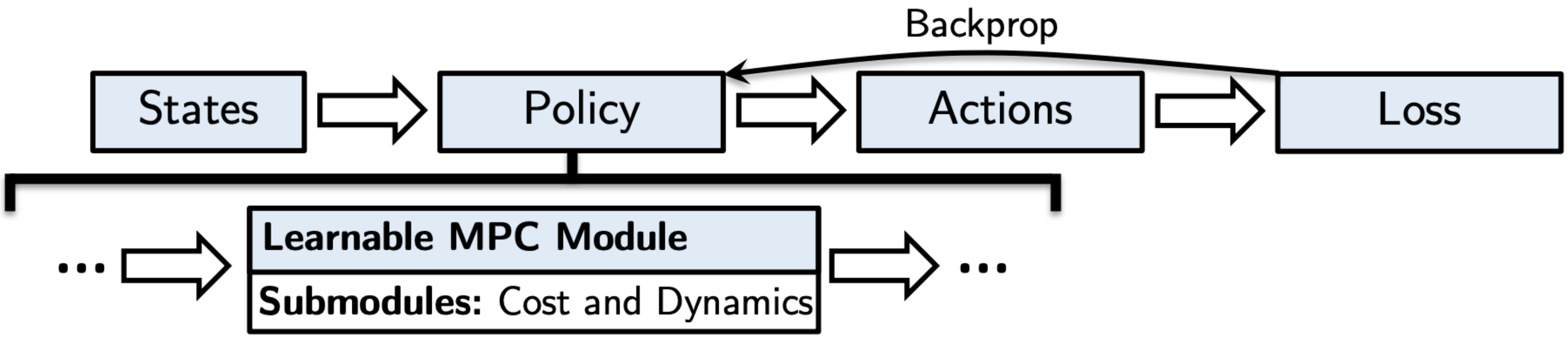
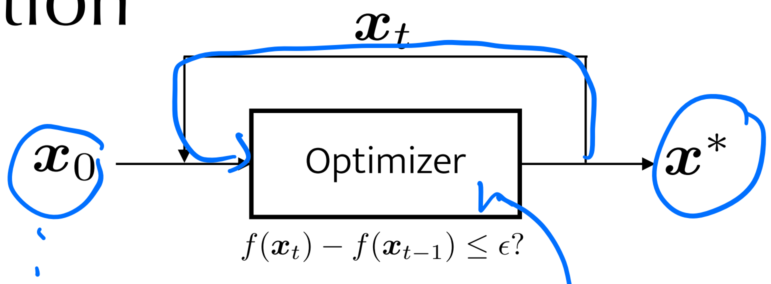
Backprop through Planning

- Treat planning as an end-to-end layer. Can be used for RL/Imitation.
- Option 1: Unrolling a finite number of steps
- Option 2: Solve till convergence, backprop for a finite step
- Option 3: Converged at fixed point: Implicit differentiation



Amos et al. Differentiable MPC for End-to-end Planning and Control. NeurIPS 2018.

Implicit Differentiation



- Unconstrained case

$$\boldsymbol{x}^* = \operatorname*{argmin}_{\boldsymbol{x}} f(\boldsymbol{x}; \boldsymbol{\theta}).$$

Cost.

Liao et al. Reviving and Improving Recurrent Back-Propagation. ICML 2018.

Implicit Differentiation

- Unconstrained case

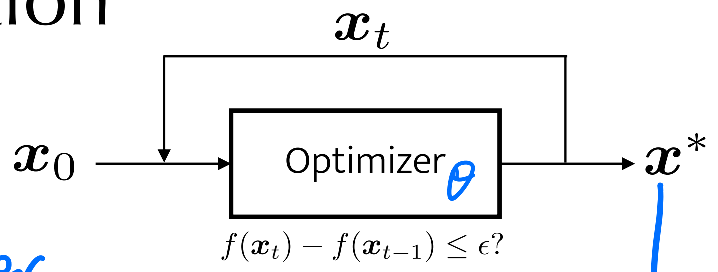
cost

$$\boldsymbol{x}^* = \arg\min_{\boldsymbol{x}} f(\boldsymbol{x};\boldsymbol{\theta}).$$

parameter.

$\frac{d}{d\theta}$

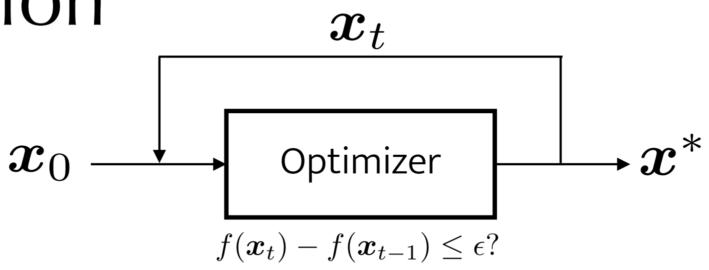
$$\mathbf{0} = \frac{\mathrm{d}}{\mathrm{d}\boldsymbol{\theta}} J_{f,\boldsymbol{x}^*}(\boldsymbol{x}^*;\boldsymbol{\theta})$$



Loss.

Liao et al. Reviving and Improving Recurrent Back-Propagation. ICML 2018.

Implicit Differentiation



- Unconstrained case

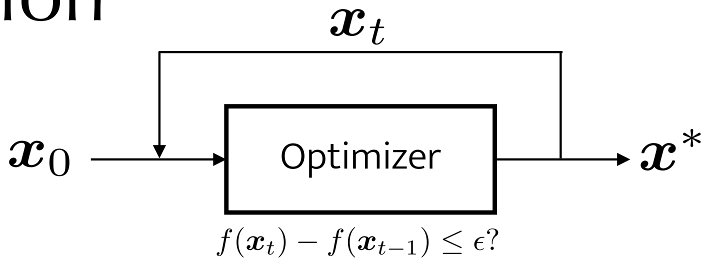
$$\boldsymbol{x}^* = \underset{\boldsymbol{x}}{\operatorname{argmin}}\, f(\boldsymbol{x}; \boldsymbol{\theta}).$$

$$\mathbf{0} = \frac{\mathrm{d}}{\mathrm{d}\boldsymbol{\theta}} J_{f,\boldsymbol{x}^*}(\boldsymbol{x}^*; \boldsymbol{\theta})$$

$$\mathbf{0} = \frac{\partial}{\partial \boldsymbol{x}^*} J_{f,\boldsymbol{x}^*}(\boldsymbol{x}^*; \boldsymbol{\theta}) \frac{\partial \boldsymbol{x}^*}{\partial \boldsymbol{\theta}} + \frac{\partial}{\partial \boldsymbol{\theta}} J_{f,\boldsymbol{x}^*}(\boldsymbol{x}^*; \boldsymbol{\theta})$$

Liao et al. Reviving and Improving Recurrent Back-Propagation. ICML 2018.

Implicit Differentiation



- Unconstrained case

$$\boldsymbol{x}^* = \underset{\boldsymbol{x}}{\operatorname{argmin}} \, f(\boldsymbol{x}; \boldsymbol{\theta}).$$

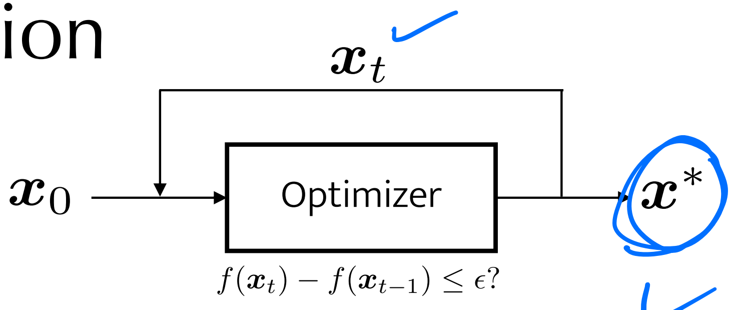
$$\mathbf{0} = \frac{\mathrm{d}}{\mathrm{d}\boldsymbol{\theta}} J_{f,\boldsymbol{x}^*}(\boldsymbol{x}^*; \boldsymbol{\theta})$$

$$\mathbf{0} = \frac{\partial}{\partial \boldsymbol{x}^*} J_{f,\boldsymbol{x}^*}(\boldsymbol{x}^*; \boldsymbol{\theta}) \frac{\partial \boldsymbol{x}^*}{\partial \boldsymbol{\theta}} + \frac{\partial}{\partial \boldsymbol{\theta}} J_{f,\boldsymbol{x}^*}(\boldsymbol{x}^*; \boldsymbol{\theta})$$

$$\mathbf{0} = H_{f,\boldsymbol{x}^*}(\boldsymbol{x}^*; \boldsymbol{\theta}) \frac{\partial \boldsymbol{x}^*}{\partial \boldsymbol{\theta}} + \frac{\partial}{\partial \boldsymbol{\theta}} J_{f,\boldsymbol{x}^*}(\boldsymbol{x}^*; \boldsymbol{\theta})$$

Liao et al. Reviving and Improving Recurrent Back-Propagation. ICML 2018.

Implicit Differentiation



- Unconstrained case

$$\boldsymbol{x}^* = \underset{\boldsymbol{x}}{\operatorname{argmin}} f(\boldsymbol{x}; \boldsymbol{\theta}).$$

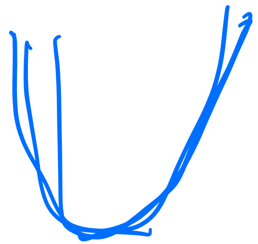
$$\mathbf{0} = \frac{\mathrm{d}}{\mathrm{d}\boldsymbol{\theta}} J_{f,\boldsymbol{x}^*}(\boldsymbol{x}^*; \boldsymbol{\theta})$$

$$\mathbf{0} = \frac{\partial}{\partial \boldsymbol{x}^*} J_{f,\boldsymbol{x}^*}(\boldsymbol{x}^*; \boldsymbol{\theta}) \frac{\partial \boldsymbol{x}^*}{\partial \boldsymbol{\theta}} + \frac{\partial}{\partial \boldsymbol{\theta}} J_{f,\boldsymbol{x}^*}(\boldsymbol{x}^*; \boldsymbol{\theta})$$

$$\mathbf{0} = H_{f,\boldsymbol{x}^*}(\boldsymbol{x}^*; \boldsymbol{\theta}) \frac{\partial \boldsymbol{x}^*}{\partial \boldsymbol{\theta}} + \frac{\partial}{\partial \boldsymbol{\theta}} J_{f,\boldsymbol{x}^*}(\boldsymbol{x}^*; \boldsymbol{\theta})$$

$$\frac{\partial \boldsymbol{x}^*}{\partial \boldsymbol{\theta}} = -\boxed{H_{f,x^*}(\boldsymbol{x}^*; \boldsymbol{\theta})}^{-1} \frac{\partial}{\partial \boldsymbol{\theta}} J_{f,x^*}(\boldsymbol{x}^*; \boldsymbol{\theta}).$$

$\boxed{N \times N}$



↳ vector.

Liao et al. Reviving and Improving Recurrent Back-Propagation. ICML 2018.

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Implicit Differentiation

- How to compute Hessian inverse vector product?

Liao et al. Reviving and Improving Recurrent Back-Propagation. ICML 2018.

Implicit Differentiation

- How to compute Hessian inverse vector product?
- Conjugate gradient, solve $\boxed{Ax = b}$

Solve x

$x = \boxed{A^{-1}b}$

Conjugate Gradient Method

$\mathbf{r}_0 := \mathbf{b} - \mathbf{A}\mathbf{x}_0$
if $\mathbf{r}_0$ is sufficiently small, then return $\mathbf{x}_0$ as the result
$\mathbf{p}_0 := \mathbf{r}_0$
$k := 0$
repeat

$$\alpha_k := \frac{\mathbf{r}_k^\top \mathbf{r}_k}{\mathbf{p}_k^\top \mathbf{A}\mathbf{p}_k}$$

$\mathbf{x}_{k+1} := \mathbf{x}_k + \alpha_k \mathbf{p}_k$
$\mathbf{r}_{k+1} := \mathbf{r}_k - \alpha_k \boxed{\mathbf{A}\mathbf{p}_k}$
if $\mathbf{r}_{k+1}$ is sufficiently small, then exit loop

$$\beta_k := \frac{\mathbf{r}_{k+1}^\top \mathbf{r}_{k+1}}{\mathbf{r}_k^\top \mathbf{r}_k}$$

$\mathbf{p}_{k+1} := \mathbf{r}_{k+1} + \beta_k \mathbf{p}_k$
$k := k + 1$
end repeat
return $\mathbf{x}_{k+1}$ as the result

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Implicit Differentiation

Jv

- How to compute Hessian inverse vector product?
- Conjugate gradient, solve $Ax = b$
- Neumann series (finite truncation)

$$(I - A)^{-1} = \sum_{k=0}^{\infty K} A^k.$$

  - Same as backprop the last K steps (Option 2).
  - Memory savings.

Conjugate Gradient Method

$\mathbf{r}_0 := \mathbf{b} - \mathbf{A}\mathbf{x}_0$

if $\mathbf{r}_0$ is sufficiently small, then return $\mathbf{x}_0$ as the result

$\mathbf{p}_0 := \mathbf{r}_0$

$k := 0$

repeat

$$\alpha_k := \frac{\mathbf{r}_k^\top \mathbf{r}_k}{\mathbf{p}_k^\top \mathbf{A}\mathbf{p}_k}$$

$\mathbf{x}_{k+1} := \mathbf{x}_k + \alpha_k \mathbf{p}_k$

$\mathbf{r}_{k+1} := \mathbf{r}_k - \alpha_k \boxed{\mathbf{A}\mathbf{p}_k}$ Hv

if $\mathbf{r}_{k+1}$ is sufficiently small, then exit loop

$$\beta_k := \frac{\mathbf{r}_{k+1}^\top \mathbf{r}_{k+1}}{\mathbf{r}_k^\top \mathbf{r}_k}$$

$\mathbf{p}_{k+1} := \mathbf{r}_{k+1} + \beta_k \mathbf{p}_k$

$k := k + 1$

end repeat

return $\mathbf{x}_{k+1}$ as the result

Liao et al. Reviving and Improving Recurrent Back-Propagation. ICML 2018.

Differentiable LQR

- Now add linear equality constraints on the dynamics and initialization.

$$\tau_{1:T} = \{x_t, u_t\}_{1:T}$$

quadratic.

function ← θ.

$$\underset{\tau_{1:T}}{\text{argmin}} \sum_{t=1}^{T} \frac{1}{2} \tau_t^\top C_t \tau_t$$

$$\text{subject to } x_{t+1} = F_t \tau_t + f_t, x_1 = x_{\text{init}}.$$

linear

Differentiable LQR

- Now add linear equality constraints on the dynamics and initialization.

$$\tau_{1:T} = \{x_t, u_t\}_{1:T}$$

$$\operatorname*{argmin}_{\tau_{1:T}} \sum_{t=1}^{T} \frac{1}{2} \tau_t^\top C_t \tau_t$$

$$\text{subject to } x_{t+1} = F_t \tau_t + f_t, x_1 = x_{\text{init}}.$$

- Chain rule: $\frac{\partial \ell}{\partial \theta} = \frac{\partial \ell}{\partial \tau^\star_{1:T}} \frac{\partial \tau^\star_{1:T}}{\partial \theta}.$

Differentiable LQR

- Now add linear equality constraints on the dynamics and initialization.

$$\tau_{1:T} = \{x_t, u_t\}_{1:T}$$

- Chain rule: $\frac{\partial \ell}{\partial \theta} = \frac{\partial \ell}{\partial \tau^\star_{1:T}} \frac{\partial \tau^\star_{1:T}}{\partial \theta}$.

$$\underset{\tau_{1:T}}{\operatorname{argmin}} \sum_{t=1}^{T} \frac{1}{2} \tau_t^\top C_t \tau_t$$

$$\text{subject to } x_{t+1} = F_t \tau_t + f_t, x_1 = x_{\text{init}}.$$

General QP:

$$\boldsymbol{x}^* = \operatorname{argmin} \underline{\frac{1}{2} \boldsymbol{x}^\top Q \boldsymbol{x} + \boldsymbol{c}^\top \boldsymbol{x}}$$

$$\text{subject to } \underline{A\boldsymbol{x} = \boldsymbol{b}}.$$

Amos et al. Differentiable MPC for End-to-end Planning and Control. NeurIPS 2018.

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Differentiable LQR

- Now add linear equality constraints on the dynamics and initialization.

$$\tau_{1:T} = \{x_t, u_t\}_{1:T}$$

- Chain rule: $\frac{\partial \ell}{\partial \theta} = \frac{\partial \ell}{\partial \tau^\star_{1:T}} \frac{\partial \tau^\star_{1:T}}{\partial \theta}.$
- KKT:

$$\operatorname*{argmin}_{\tau_{1:T}} \sum_{t=1}^{T} \frac{1}{2} \tau_t^\top C_t \tau_t$$

$$\text{subject to } x_{t+1} = F_t \tau_t + f_t, x_1 = x_{\text{init}}.$$

General QP:

$$\boldsymbol{x}^* = \operatorname{argmin} \frac{1}{2} \boldsymbol{x}^\top Q \boldsymbol{x} + \boldsymbol{c}^\top \boldsymbol{x}$$

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Differentiable LQR

- Now add linear equality constraints on the dynamics and initialization.

$$\tau_{1:T} = \{x_t, u_t\}_{1:T}$$

- Chain rule: $\frac{\partial \ell}{\partial \theta} = \frac{\partial \ell}{\partial \tau^\star_{1:T}} \frac{\partial \tau^\star_{1:T}}{\partial \theta}.$
- KKT:

$$\underbrace{\begin{bmatrix} Q & A^\top \\ A & \mathbf{0} \end{bmatrix}}_{=\,K} \begin{bmatrix} \boldsymbol{x}^* \\ \boldsymbol{\lambda}^* \end{bmatrix} = \begin{bmatrix} -\boldsymbol{c} \\ \boldsymbol{b} \end{bmatrix}$$

$$K \begin{bmatrix} \boldsymbol{x}^* \\ \boldsymbol{\lambda}^* \end{bmatrix} = v.$$

$$\underset{\tau_{1:T}}{\operatorname{argmin}} \sum_{t=1}^{T} \frac{1}{2} \tau_t^\top C_t \tau_t$$

$$\text{subject to } x_{t+1} = F_t \tau_t + f_t, x_1 = x_{\text{init}}.$$

General QP:

$$\boldsymbol{x}^* = \operatorname{argmin} \frac{1}{2} \boldsymbol{x}^\top Q \boldsymbol{x} + \boldsymbol{c}^\top \boldsymbol{x}$$

$$\text{subject to } \underbrace{A\boldsymbol{x} = \boldsymbol{b}}.$$

In classic LQR solver, the Riccati recursion solves this linear system.

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Differentiable LQR

- Apply differentiation

$$\frac{\mathrm{d}}{\mathrm{d}\boldsymbol{\theta}}\left(K\begin{bmatrix}\boldsymbol{x}^*\\ \boldsymbol{\lambda}^*\end{bmatrix}\right)=\frac{\mathrm{d}v}{\mathrm{d}\boldsymbol{\theta}}.$$

$$\frac{\mathrm{d}K}{\mathrm{d}\boldsymbol{\theta}}\begin{bmatrix}\boldsymbol{x}^*\\ \boldsymbol{\lambda}^*\end{bmatrix}+K\begin{bmatrix}\frac{\mathrm{d}\boldsymbol{x}^*}{\mathrm{d}\boldsymbol{\theta}}\\ \frac{\mathrm{d}\boldsymbol{\lambda}^*}{\mathrm{d}\boldsymbol{\theta}}\end{bmatrix}=\frac{\mathrm{d}v}{\mathrm{d}\boldsymbol{\theta}}$$

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Differentiable LQR

- Apply differentiation

$$\frac{\mathrm{d}}{\mathrm{d}\boldsymbol{\theta}}\left(K\begin{bmatrix}\boldsymbol{x}^*\\ \boldsymbol{\lambda}^*\end{bmatrix}\right)=\frac{\mathrm{d}v}{\mathrm{d}\boldsymbol{\theta}}.$$

$$\frac{\mathrm{d}K}{\mathrm{d}\boldsymbol{\theta}}\begin{bmatrix}\boldsymbol{x}^*\\ \boldsymbol{\lambda}^*\end{bmatrix}+K\begin{bmatrix}\frac{\mathrm{d}\boldsymbol{x}^*}{\mathrm{d}\boldsymbol{\theta}}\\ \frac{\mathrm{d}\boldsymbol{\lambda}^*}{\mathrm{d}\boldsymbol{\theta}}\end{bmatrix}=\frac{\mathrm{d}v}{\mathrm{d}\boldsymbol{\theta}}$$

$$K\begin{bmatrix}\frac{\mathrm{d}\boldsymbol{x}^*}{\mathrm{d}\boldsymbol{\theta}}\\ \frac{\mathrm{d}\boldsymbol{\lambda}^*}{\mathrm{d}\boldsymbol{\theta}}\end{bmatrix}=K\begin{bmatrix}\frac{\mathrm{d}\boldsymbol{x}^*}{\mathrm{d}\boldsymbol{c}} & \frac{\mathrm{d}\boldsymbol{x}^*}{\mathrm{d}\boldsymbol{b}} & \frac{\mathrm{d}\boldsymbol{x}^*}{\mathrm{d}Q} & \frac{\mathrm{d}\boldsymbol{x}^*}{\mathrm{d}A}\\ \frac{\mathrm{d}\boldsymbol{\lambda}^*}{\mathrm{d}\boldsymbol{c}} & \frac{\mathrm{d}\boldsymbol{\lambda}^*}{\mathrm{d}\boldsymbol{b}} & \frac{\mathrm{d}\boldsymbol{\lambda}^*}{\mathrm{d}Q} & \frac{\mathrm{d}\boldsymbol{\lambda}^*}{\mathrm{d}A}\end{bmatrix}=\begin{bmatrix}-I & 0 & -x^* & -\lambda^*\\ 0 & I & 0 & -x^*\end{bmatrix}$$

Amos et al. *Differentiable MPC for End-to-end Planning and Control. NeurIPS 2018.*

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Differentiable LQR

- Apply differentiation

$$\frac{\mathrm{d}}{\mathrm{d}\boldsymbol{\theta}}\left(K\begin{bmatrix}\boldsymbol{x}^*\\ \boldsymbol{\lambda}^*\end{bmatrix}\right)=\frac{\mathrm{d}v}{\mathrm{d}\boldsymbol{\theta}}.$$

$$\frac{\mathrm{d}K}{\mathrm{d}\boldsymbol{\theta}}\begin{bmatrix}\boldsymbol{x}^*\\ \boldsymbol{\lambda}^*\end{bmatrix}+K\begin{bmatrix}\frac{\mathrm{d}\boldsymbol{x}^*}{\mathrm{d}\boldsymbol{\theta}}\\ \frac{\mathrm{d}\boldsymbol{\lambda}^*}{\mathrm{d}\boldsymbol{\theta}}\end{bmatrix}=\frac{\mathrm{d}v}{\mathrm{d}\boldsymbol{\theta}}$$

$$K\begin{bmatrix}\frac{\mathrm{d}\boldsymbol{x}^*}{\mathrm{d}\boldsymbol{\theta}}\\ \frac{\mathrm{d}\boldsymbol{\lambda}^*}{\mathrm{d}\boldsymbol{\theta}}\end{bmatrix}=K\begin{bmatrix}\frac{\mathrm{d}\boldsymbol{x}^*}{\mathrm{d}\boldsymbol{c}} & \frac{\mathrm{d}\boldsymbol{x}^*}{\mathrm{d}\boldsymbol{b}} & \frac{\mathrm{d}\boldsymbol{x}^*}{\mathrm{d}Q} & \frac{\mathrm{d}\boldsymbol{x}^*}{\mathrm{d}A}\\ \frac{\mathrm{d}\boldsymbol{\lambda}^*}{\mathrm{d}\boldsymbol{c}} & \frac{\mathrm{d}\boldsymbol{\lambda}^*}{\mathrm{d}\boldsymbol{b}} & \frac{\mathrm{d}\boldsymbol{\lambda}^*}{\mathrm{d}Q} & \frac{\mathrm{d}\boldsymbol{\lambda}^*}{\mathrm{d}A}\end{bmatrix}=\begin{bmatrix}-I & 0 & -x^* & -\lambda^*\\ 0 & I & 0 & -x^*\end{bmatrix}$$

$$K\frac{\partial \ell}{\partial \boldsymbol{z}^*}\begin{bmatrix}\frac{\mathrm{d}\boldsymbol{x}^*}{\mathrm{d}\boldsymbol{c}} & \frac{\mathrm{d}\boldsymbol{x}^*}{\mathrm{d}\boldsymbol{b}}\\ \frac{\mathrm{d}\boldsymbol{\lambda}^*}{\mathrm{d}\boldsymbol{c}} & \frac{\mathrm{d}\boldsymbol{\lambda}^*}{\mathrm{d}\boldsymbol{b}}\end{bmatrix}=\begin{bmatrix}-\frac{\partial \ell}{\partial \boldsymbol{x}^*}\\ \mathbf{0}\end{bmatrix}$$

$$K\boldsymbol{d}^*=\begin{bmatrix}-\frac{\partial \ell}{\partial \boldsymbol{x}^*}\\ \mathbf{0}\end{bmatrix}$$

Amos et al. Differentiable MPC for End-to-end Planning and Control. NeurIPS 2018.

Differentiable LQR

- Apply differentiation

$$\frac{\mathrm{d}}{\mathrm{d}\boldsymbol{\theta}}\left(K\begin{bmatrix}\boldsymbol{x}^*\\ \boldsymbol{\lambda}^*\end{bmatrix}\right) = \frac{\mathrm{d}v}{\mathrm{d}\boldsymbol{\theta}}.$$

$$\frac{\mathrm{d}K}{\mathrm{d}\boldsymbol{\theta}}\begin{bmatrix}\boldsymbol{x}^*\\ \boldsymbol{\lambda}^*\end{bmatrix} + K\begin{bmatrix}\frac{\mathrm{d}\boldsymbol{x}^*}{\mathrm{d}\boldsymbol{\theta}}\\ \frac{\mathrm{d}\boldsymbol{\lambda}^*}{\mathrm{d}\boldsymbol{\theta}}\end{bmatrix} = \frac{\mathrm{d}v}{\mathrm{d}\boldsymbol{\theta}}$$

$$K\begin{bmatrix}\frac{\mathrm{d}\boldsymbol{x}^*}{\mathrm{d}\boldsymbol{\theta}}\\ \frac{\mathrm{d}\boldsymbol{\lambda}^*}{\mathrm{d}\boldsymbol{\theta}}\end{bmatrix} = K\begin{bmatrix}\frac{\mathrm{d}\boldsymbol{x}^*}{\mathrm{d}\boldsymbol{c}} & \frac{\mathrm{d}\boldsymbol{x}^*}{\mathrm{d}\boldsymbol{b}} & \frac{\mathrm{d}\boldsymbol{x}^*}{\mathrm{d}Q} & \frac{\mathrm{d}\boldsymbol{x}^*}{\mathrm{d}A}\\ \frac{\mathrm{d}\boldsymbol{\lambda}^*}{\mathrm{d}\boldsymbol{c}} & \frac{\mathrm{d}\boldsymbol{\lambda}^*}{\mathrm{d}\boldsymbol{b}} & \frac{\mathrm{d}\boldsymbol{\lambda}^*}{\mathrm{d}Q} & \frac{\mathrm{d}\boldsymbol{\lambda}^*}{\mathrm{d}A}\end{bmatrix} = \begin{bmatrix}-I & 0 & -x^* & -\lambda^*\\ 0 & I & 0 & -x^*\end{bmatrix} \qquad K\frac{\partial \ell}{\partial \boldsymbol{z}^*}\begin{bmatrix}\frac{\mathrm{d}\boldsymbol{x}^*}{\mathrm{d}\boldsymbol{c}} & \frac{\mathrm{d}\boldsymbol{x}^*}{\mathrm{d}\boldsymbol{b}}\\ \frac{\mathrm{d}\boldsymbol{\lambda}^*}{\mathrm{d}\boldsymbol{c}} & \frac{\mathrm{d}\boldsymbol{\lambda}^*}{\mathrm{d}\boldsymbol{b}}\end{bmatrix} = \begin{bmatrix}-\frac{\partial \ell}{\partial \boldsymbol{x}^*}\\ \boldsymbol{0}\end{bmatrix}$$

$$\underbrace{K\boldsymbol{d}^* = \begin{bmatrix}-\frac{\partial \ell}{\partial \boldsymbol{x}^*}\\ \boldsymbol{0}\end{bmatrix}}$$

- Equivalent QP:

$$\left.\begin{aligned}\boldsymbol{d}^* = \underset{\boldsymbol{d}}{\operatorname{argmin}}\ & \frac{1}{2}\boldsymbol{d}^\top Q\boldsymbol{d} + \frac{\partial \ell}{\partial \boldsymbol{x}^*}^\top \boldsymbol{d},\\ \text{subject to}\ & A\boldsymbol{d} = \boldsymbol{0}.\end{aligned}\right\} \to \underline{LQR}$$

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Differentiable LQR

- Apply differentiation

$$\frac{\mathrm{d}}{\mathrm{d}\boldsymbol{\theta}}\left(K\begin{bmatrix}\boldsymbol{x}^*\\ \boldsymbol{\lambda}^*\end{bmatrix}\right)=\frac{\mathrm{d}v}{\mathrm{d}\boldsymbol{\theta}}.$$

$$\frac{\mathrm{d}K}{\mathrm{d}\boldsymbol{\theta}}\begin{bmatrix}\boldsymbol{x}^*\\ \boldsymbol{\lambda}^*\end{bmatrix}+K\begin{bmatrix}\frac{\mathrm{d}\boldsymbol{x}^*}{\mathrm{d}\boldsymbol{\theta}}\\ \frac{\mathrm{d}\boldsymbol{\lambda}^*}{\mathrm{d}\boldsymbol{\theta}}\end{bmatrix}=\frac{\mathrm{d}v}{\mathrm{d}\boldsymbol{\theta}}$$

$$K\begin{bmatrix}\frac{\mathrm{d}\boldsymbol{x}^*}{\mathrm{d}\boldsymbol{\theta}}\\ \frac{\mathrm{d}\boldsymbol{\lambda}^*}{\mathrm{d}\boldsymbol{\theta}}\end{bmatrix}=K\begin{bmatrix}\frac{\mathrm{d}\boldsymbol{x}^*}{\mathrm{d}\boldsymbol{c}} & \frac{\mathrm{d}\boldsymbol{x}^*}{\mathrm{d}\boldsymbol{b}} & \frac{\mathrm{d}\boldsymbol{x}^*}{\mathrm{d}Q} & \frac{\mathrm{d}\boldsymbol{x}^*}{\mathrm{d}A}\\ \frac{\mathrm{d}\boldsymbol{\lambda}^*}{\mathrm{d}\boldsymbol{c}} & \frac{\mathrm{d}\boldsymbol{\lambda}^*}{\mathrm{d}\boldsymbol{b}} & \frac{\mathrm{d}\boldsymbol{\lambda}^*}{\mathrm{d}Q} & \frac{\mathrm{d}\boldsymbol{\lambda}^*}{\mathrm{d}A}\end{bmatrix}=\begin{bmatrix}-I & 0 & -x^* & -\lambda^*\\ 0 & I & 0 & -x^*\end{bmatrix}\quad K\frac{\partial \ell}{\partial \boldsymbol{z}^*}\begin{bmatrix}\frac{\mathrm{d}\boldsymbol{x}^*}{\mathrm{d}\boldsymbol{c}} & \frac{\mathrm{d}\boldsymbol{x}^*}{\mathrm{d}\boldsymbol{b}}\\ \frac{\mathrm{d}\boldsymbol{\lambda}^*}{\mathrm{d}\boldsymbol{c}} & \frac{\mathrm{d}\boldsymbol{\lambda}^*}{\mathrm{d}\boldsymbol{b}}\end{bmatrix}=\begin{bmatrix}-\frac{\partial \ell}{\partial \boldsymbol{x}^*}\\ \boldsymbol{0}\end{bmatrix}$$

$$K\boldsymbol{d}^*=\begin{bmatrix}-\frac{\partial \ell}{\partial \boldsymbol{x}^*}\\ \boldsymbol{0}\end{bmatrix}$$

- Equivalent QP:

$$\boldsymbol{d}^*=\underset{\boldsymbol{d}}{\operatorname{argmin}}\ \frac{1}{2}\boldsymbol{d}^\top Q\boldsymbol{d}+\frac{\partial \ell}{\partial \boldsymbol{x}^*}^\top\boldsymbol{d},\quad \text{subject to}\ \ A\boldsymbol{d}=\boldsymbol{0}.$$

$$\frac{\partial \ell}{\partial Q}=\frac{1}{2}(\underbrace{\boldsymbol{d}_{\boldsymbol{x}}^*\otimes\boldsymbol{x}^*}+\boldsymbol{x}^*\otimes\boldsymbol{d}_{\boldsymbol{x}}^*)$$

$$\frac{\partial \ell}{\partial A}=\boldsymbol{d}_{\boldsymbol{\lambda}}^*\otimes\underline{\boldsymbol{x}^*}+\underline{\boldsymbol{\lambda}^*}\otimes\boldsymbol{d}_{\boldsymbol{x}}^*.$$

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Differentiable LQR

- The backward pass can also be formulated as a LQR problem.
- Swap c to $\nabla_{\tau^\star}\ell$ and f to 0.

Module 1 Differentiable LQR *(The LQR algorithm is defined in appendix A)*

Input: Initial state x_{init}
Parameters: $\theta = \{C, c, F, f\}$

Forward Pass:

1: $\tau^\star_{1:T} = \text{LQR}_T(x_{\text{init}}; C, c, F, f)$ ▷ Solve (2)
2: Compute $\lambda^\star_{1:T}$ with (7)

Backward Pass:

1: $d^\star_{\tau_{1:T}} = \text{LQR}_T(0; C, \nabla_{\tau^\star}\ell, F, 0)$ ▷ Solve (9), ideally reusing the factorizations from the forward pass
2: Compute $d^\star_{\lambda_{1:T}}$ with (7)
3: Compute the derivatives of ℓ with respect to C, c, F, f, and x_{init} with (8)

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Differentiable MPC

- What about general MPC?

$$\underset{x_{1:T}\in\mathcal{X},u_{1:T}\in\mathcal{U}}{\operatorname{argmin}} \sum_{t=1}^{T} \boxed{C_t(x_t, u_t)}$$
$$\text{subject to } x_{t+1} = f(x_t, u_t), x_1 = x_{\text{init}}.$$

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Differentiable MPC

- What about general MPC?

$$\underset{x_{1:T}\in\mathcal{X},u_{1:T}\in\mathcal{U}}{\operatorname{argmin}} \sum_{t=1}^{T} C_t(x_t, u_t)$$

$$\text{subject to } x_{t+1} = f(x_t, u_t), x_1 = x_{\text{init}}.$$

- Use Taylor expansion to approximate.

$$\tilde{C}^i_{\theta,t} = C_{\theta,t}(\tau^i_t) + {p^i_t}^\top(\tau_t - \tau^i_t) + \frac{1}{2}(\tau_t - \tau^i_t)^\top H^i_t(\tau_t - \tau^i_t).$$

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Differentiable MPC

- What about general MPC?

$$\underset{x_{1:T}\in\mathcal{X},u_{1:T}\in\mathcal{U}}{\operatorname{argmin}} \sum_{t=1}^{T} C_t(x_t, u_t)$$

$$\text{subject to } x_{t+1} = f(x_t, u_t), x_1 = x_{\text{init}}.$$

- Use Taylor expansion to approximate.

$$\tilde{C}_{\theta,t}^{i} = C_{\theta,t}(\tau_t^i) + {p_t^i}^\top(\tau_t - \tau_t^i) + \frac{1}{2}(\tau_t - \tau_t^i)^\top H_t^i(\tau_t - \tau_t^i).$$

- Fixed point iteration.

$$\tau^{i+1} = \operatorname{argmin}_\tau \textstyle\sum_t^T \tilde{C}_t^i(\tau_t^i).$$

Amos et al. *Differentiable MPC for End-to-end Planning and Control*. NeurIPS 2018.

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Differentiable MPC

- What about general MPC?

$$\operatorname*{argmin}_{x_{1:T}\in\mathcal{X},u_{1:T}\in\mathcal{U}} \sum_{t=1}^{T} C_t(x_t,u_t)$$

$$\text{subject to } x_{t+1} = f(x_t,u_t), x_1 = x_{\text{init}}.$$

- Use Taylor expansion to approximate.

$$\tilde{C}^i_{\theta,t} = C_{\theta,t}(\tau^i_t) + {p^i_t}^\top(\tau_t - \tau^i_t) + \frac{1}{2}(\tau_t - \tau^i_t)^\top H^i_t(\tau_t - \tau^i_t).$$

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- Backward only depends on final quadratic approximation.

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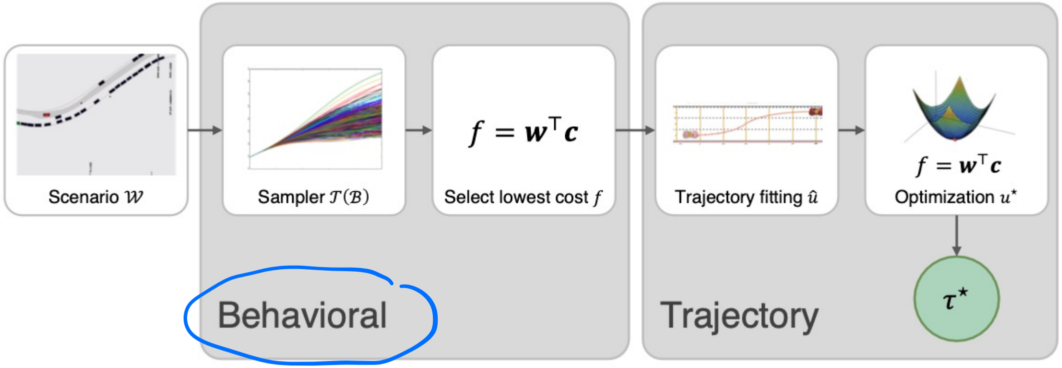
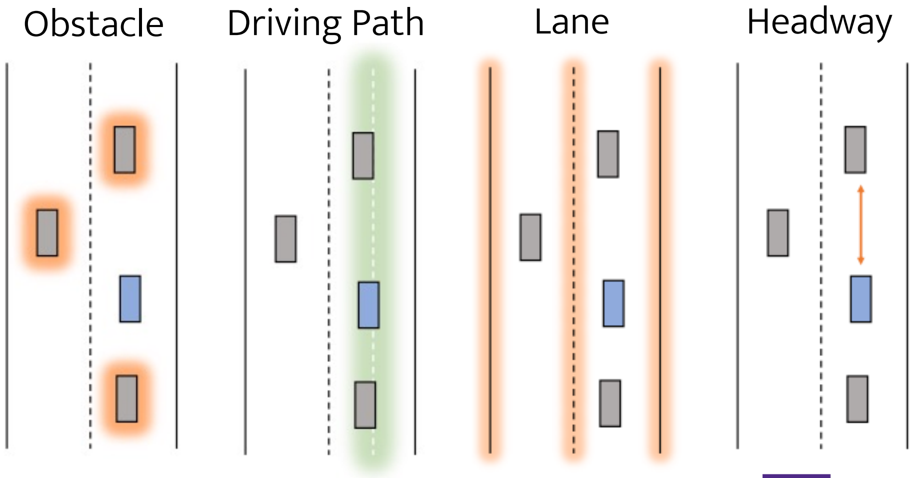
Behavioral vs. Trajectory Planning

- Gradient-based optimization provides a locally optimized trajectory.





Sadat et al. Jointly Learnable Behavior and Trajectory Planning for Self-Driving Vehicles. IROS 2019.

Behavioral vs. Trajectory Planning

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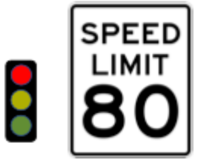

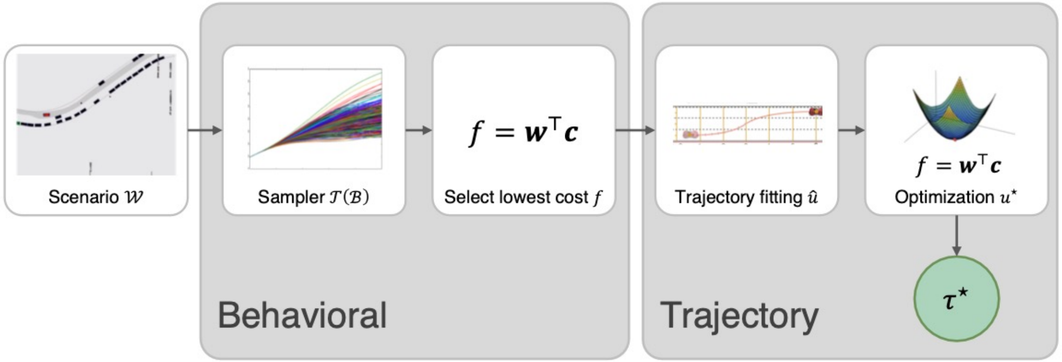

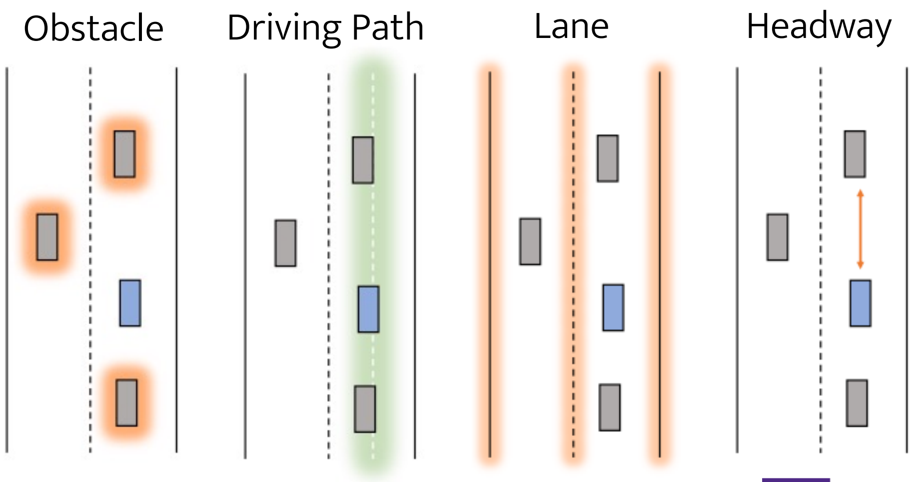

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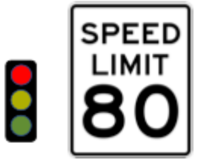
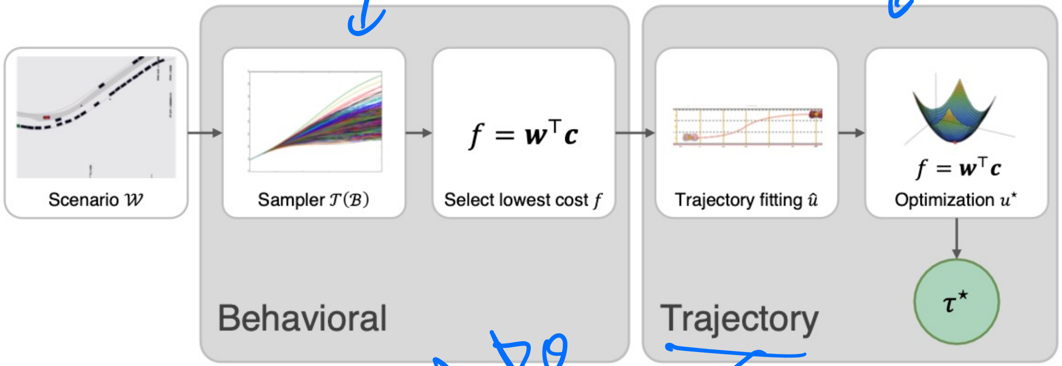
Behavioral vs. Trajectory Planning

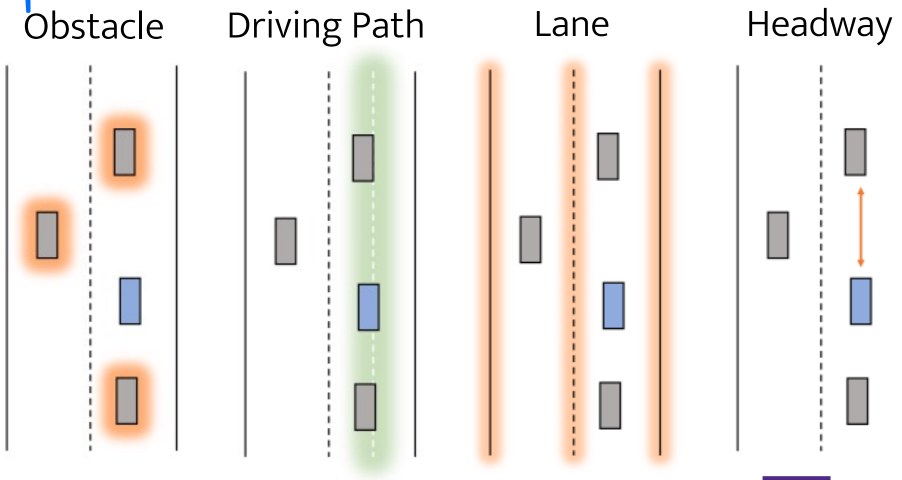
- Gradient-based optimization provides a locally optimized trajectory.
- Samples may be needed for reasoning global structure.
- Can learn together using the same learned costs.



max-margin

backprop.





$\nabla\theta$

$\nabla\theta$

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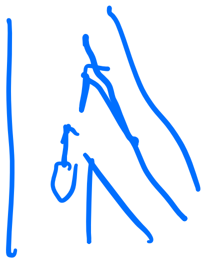
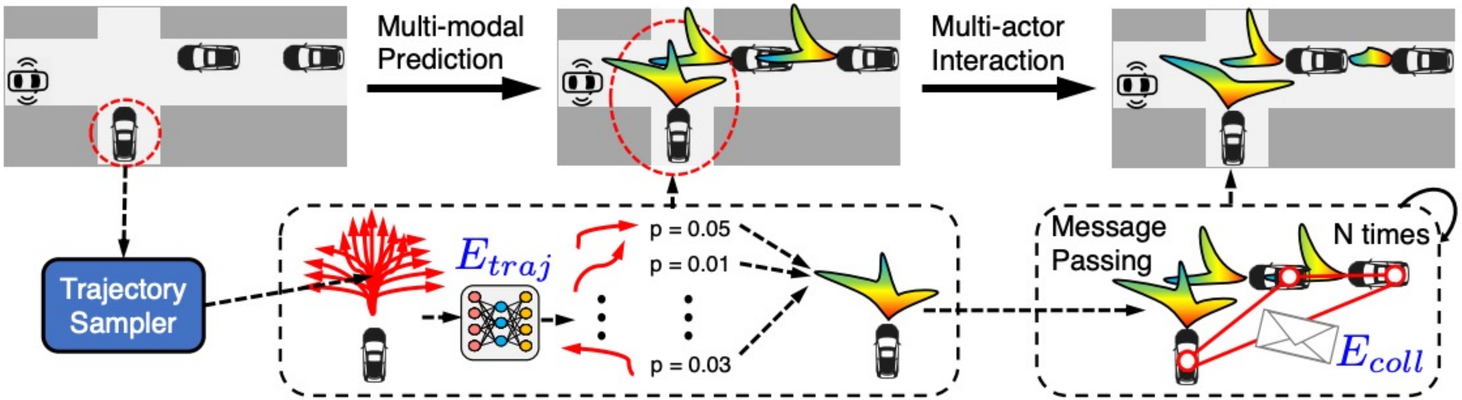
Planning with Social Reasoning



- Jointly reason the future trajectories of multiple agents as an energy-based graphical model.

$$p(\mathbf{s}_1, \ldots, \mathbf{s}_N \mid \mathbf{X}) = \frac{1}{Z} \exp(-E_\theta(\mathbf{s}_1, \ldots, \mathbf{s}_N) \mid \mathbf{X})$$



Zeng et al. DSDNet: Deep Structured Self-Driving Network. ECCV 2020.

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Planning with Social Reasoning

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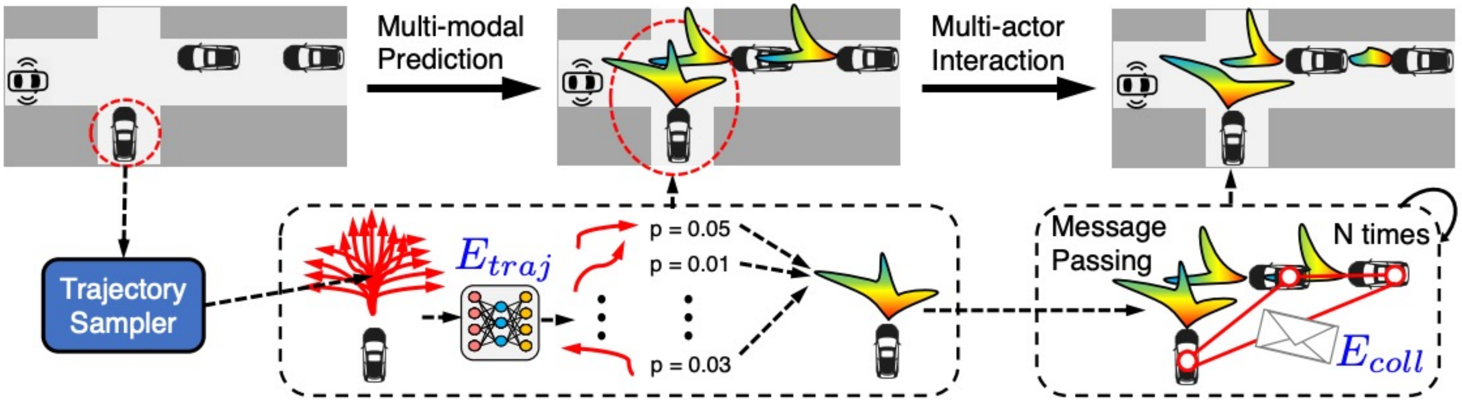
$$\sum_i E_\theta(s_i \mid X) + \sum_{i \neq j} \underbrace{E(\mathbf{s}_i, \mathbf{s}_j)}$$

$$\underbrace{E(\mathbf{s}_i, \mathbf{s}_j)} = \boxed{\gamma} \text{ if } \mathbf{s}_i \text{ collides } \mathbf{s}_j$$



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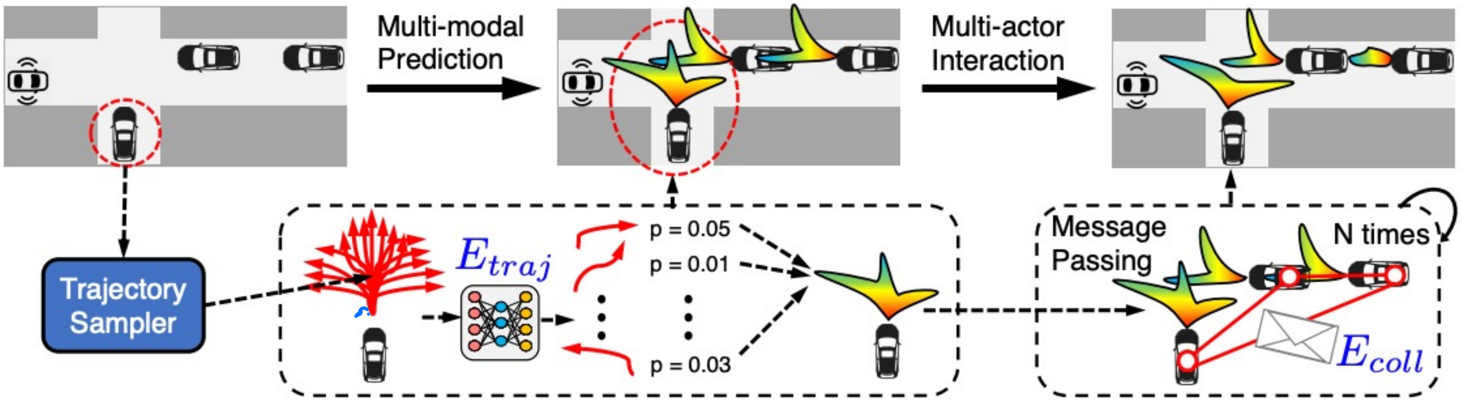

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Planning with Social Reasoning

- Jointly reason the future trajectories of multiple agents as an energy-based graphical model.
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- Batch of trajectory samples.
- Classification of groundtruth trajectory.

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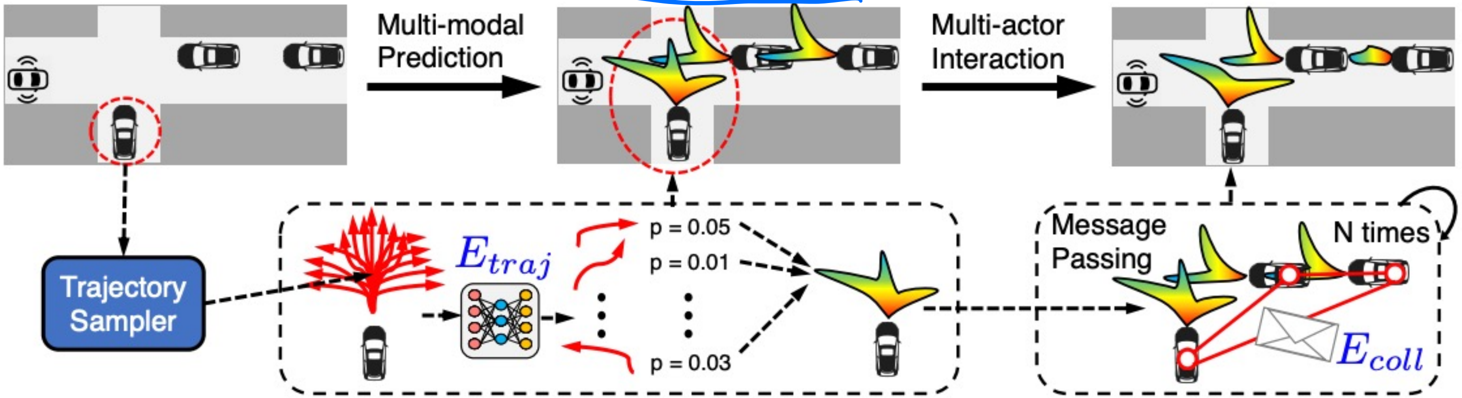

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Summary: End-to-End Planning

- Direct Policy Prediction
  - Condition perception features into the model
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- Differentiable Planner
  - Backprop through local optimization
  - Can be memory efficient, implicit differentiation

Module 5:
Continual Learning,
Few-Shot Learning, Meta-Learning

Why Continual Learning?

- The world is not a dataset that allows you to get IID samples.

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- The world is not a dataset that allows you to get IID samples.
- The world keeps changing and evolving.
- Online vs. Continual
  - Online means that samples arrive in a streaming / temporal partial order, but they may still come from a static distribution.

$$\theta_t = f(x_t, \theta_{t-1}) \quad x_{1:T} \sim \mathcal{X}$$

  - Example: Online reinforcement learning, trajectory roll out is online, but the environment is the same.
  - Continual learning means that there will be distribution shift.

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What is Continual Learning?

- Distribution shift: Forgetting
  - Learning on A and then B, results in worse performance on A.

backward transfer.

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- Compositionality
  - Learning A and B first, and then learning tasks with composed A+B.
- Incremental/curriculum Learning
  - Learning A->B->C is easier than at random order.

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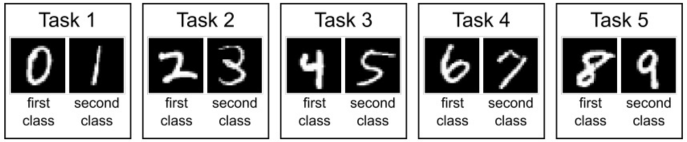
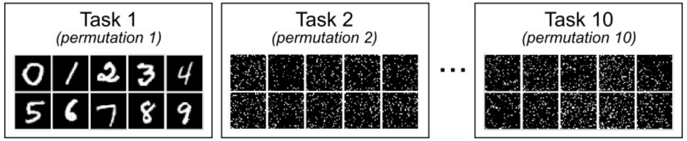
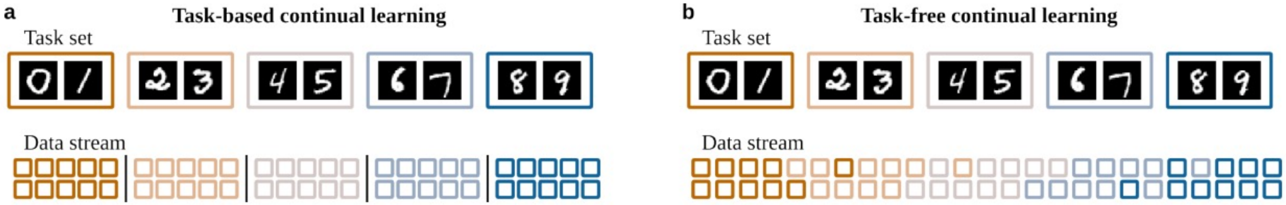
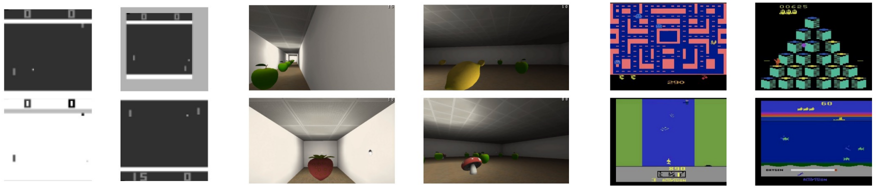
Continual Learning

- Learning a sequence of tasks without looking back.

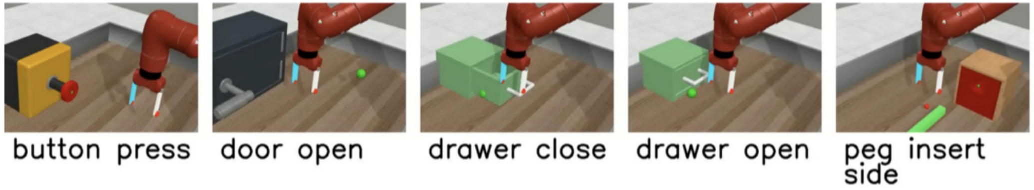








(a) Pong variants
(b) Labyrinth games
(c) Atari games

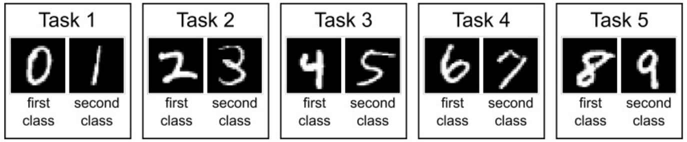


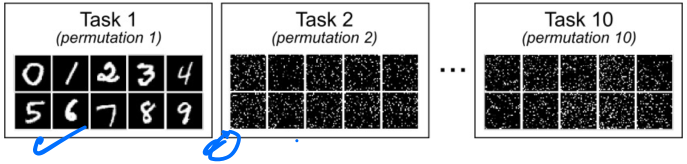
van de Ven et al. Continual learning and catastrophic forgetting. arXiv 2024.
van de Ven & Tolias. Three scenarios for continual learning. arXiv 2019.

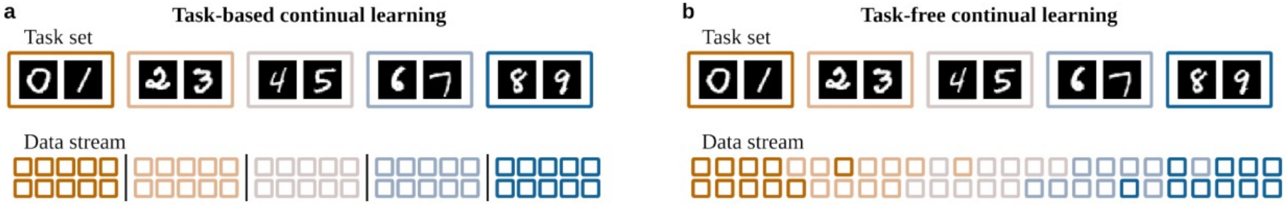
Continual Learning

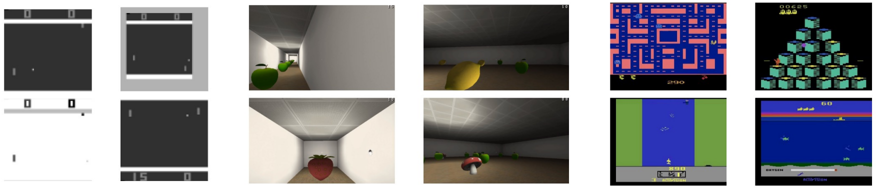
increment

- Learning a sequence of tasks without looking back.
- Goal is to do well on all of the tasks at the end.









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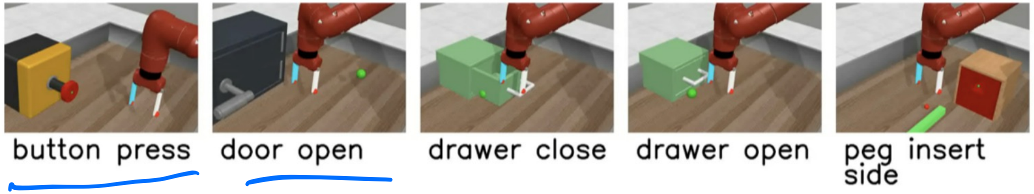

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Continual Learning

- Learning a sequence of tasks without looking back.
- Goal is to do well on all of the tasks at the end.
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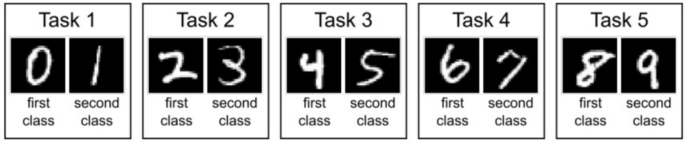

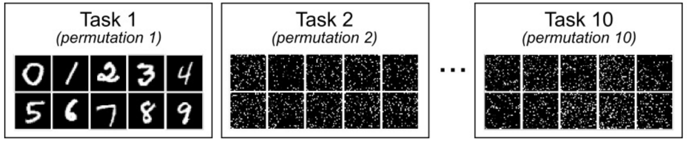

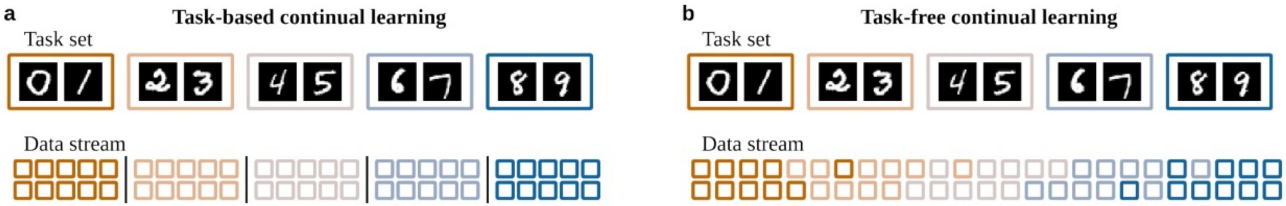

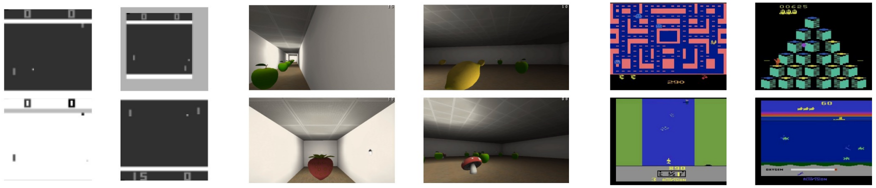

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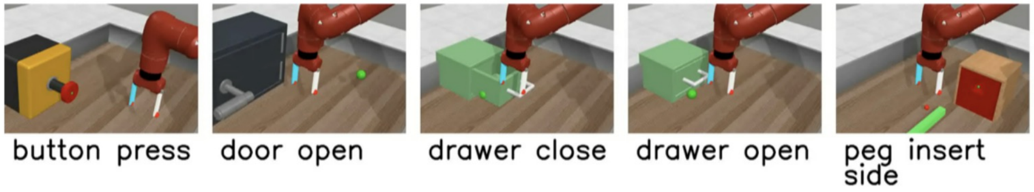

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Continual Learning

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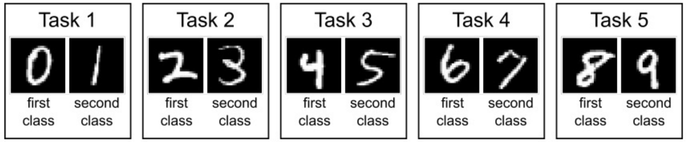
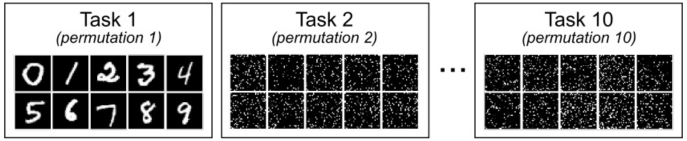
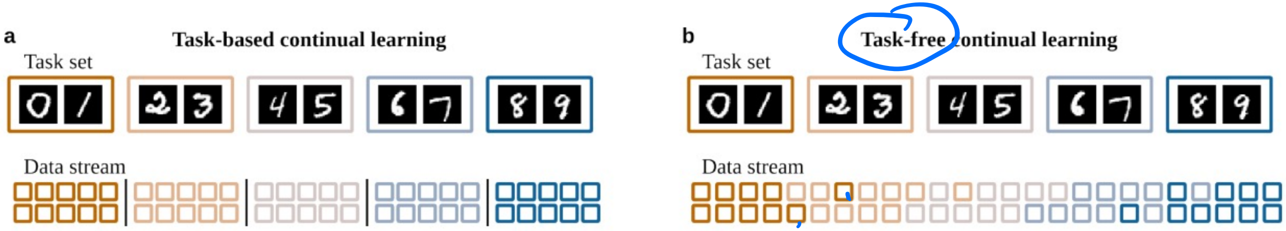
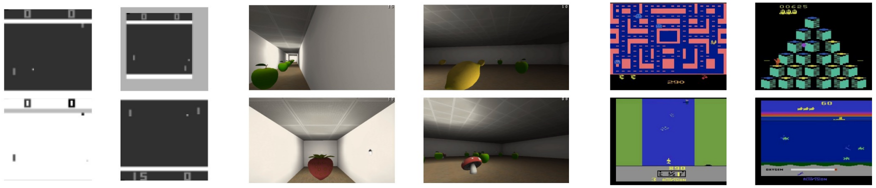
- Memory constraints









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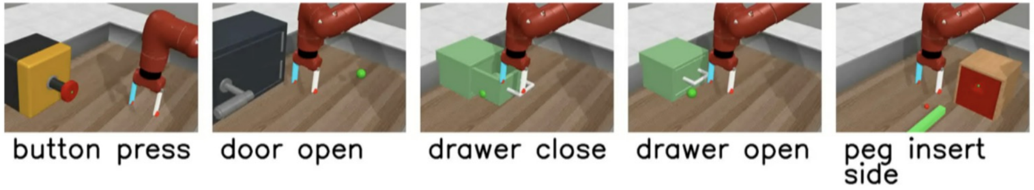

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Parameter Regularization

- Over-completeness Assumption. A multitude of models can reach equivalent performance.

$$\mathcal{S}_A = \{\theta \mid \ell_A(\theta) < \epsilon\}$$

$$\mathcal{S}_A \cap \mathcal{S}_B \neq \emptyset$$

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Parameter Regularization

- Over-completeness Assumption. A multitude of models can reach equivalent performance.
- What is left is to efficiently find the intersection between A and B.

$$p(\theta \mid \mathcal{D}_A) = \mathcal{N}(\theta; \theta^*, \Sigma)$$

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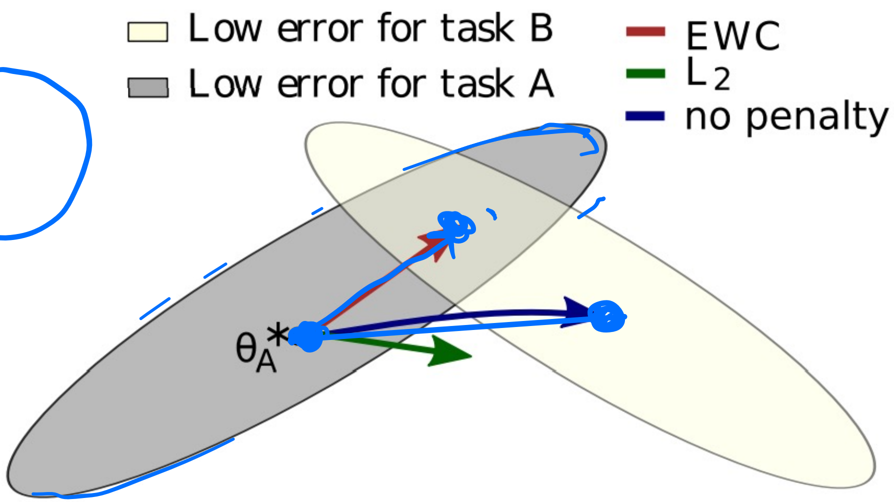
$$\mathcal{L}(\theta) = \mathcal{L}_B(\theta) + \sum_i \frac{\lambda}{2} F_i (\theta_i - \theta^*_{A,i})^2$$



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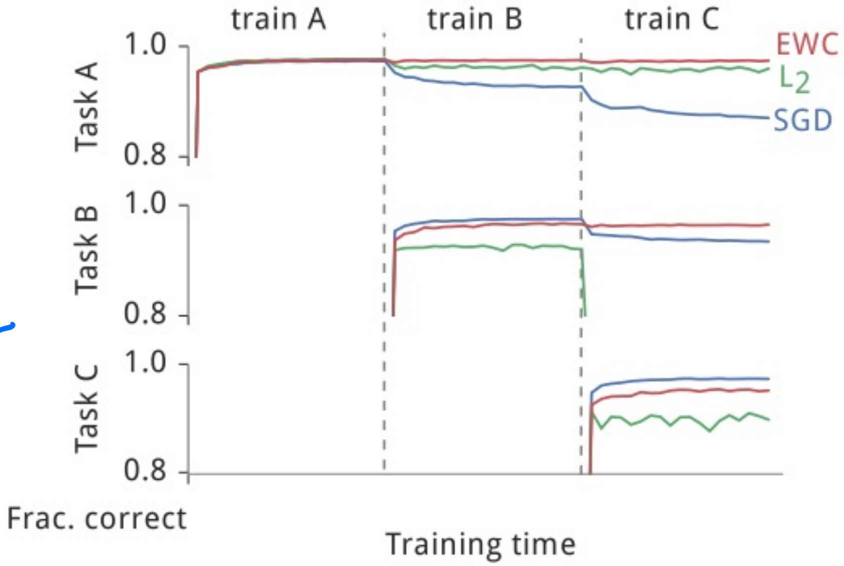

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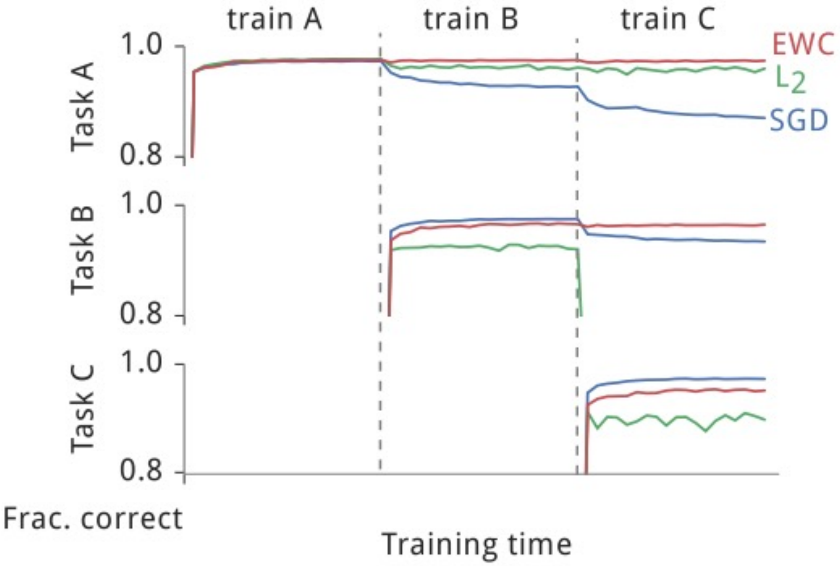

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Computing Fisher

plasticity --
stability. ++

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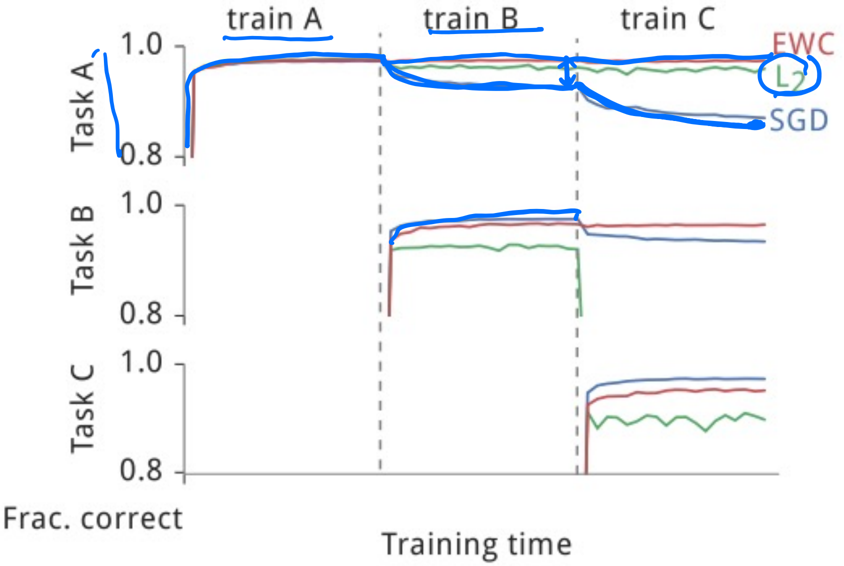
- Measures the sensitivity on each parameter dimension.
- You can also accumulate an online estimate.



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Variational Continual Learning (VCL)

- Bayesian formulation:

$$p(\boldsymbol{\theta} \mid \mathcal{D}_{1:T}) \propto p(\boldsymbol{\theta}) \prod_{t=1}^{T} p(\mathcal{D}_t \mid \boldsymbol{\theta}) \propto p(\boldsymbol{\theta} \mid \mathcal{D}_{1:T-1}) p(\mathcal{D}_T \mid \boldsymbol{\theta}).$$

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$$q_t(\boldsymbol{\theta}) = \prod_{d=1}^{D} \mathcal{N}(\theta_{t,d}; \mu_{t,d}, \sigma^2_{t,d}).$$

Nguyen et al. Variational Continual Learning. ICLR 2018.

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Variational Continual Learning (VCL)

- Bayesian formulation:

$$p(\boldsymbol{\theta} \mid \mathcal{D}_{1:T}) \propto p(\boldsymbol{\theta}) \prod_{t=1}^{T} p(\mathcal{D}_t \mid \boldsymbol{\theta}) \propto p(\boldsymbol{\theta} \mid \mathcal{D}_{1:T-1}) p(\mathcal{D}_T \mid \boldsymbol{\theta}).$$

- Variational approach:

$$q_t(\boldsymbol{\theta}) = \operatorname*{argmin}_{q \in \mathcal{Q}} \mathrm{KL}\left(q(\boldsymbol{\theta}) \parallel \frac{1}{Z_t} q_{t-1}(\boldsymbol{\theta}) p(\mathcal{D}_t \mid \boldsymbol{\theta}) \right).$$

- Loss: $\mathcal{L}(q_t(\boldsymbol{\theta})) = \mathbb{E}_{\boldsymbol{\theta} \sim q_t(\boldsymbol{\theta})}[-\log p(\mathbf{y}|\mathbf{x}, \theta)] + \mathrm{KL}(q_t(\boldsymbol{\theta} \parallel q_{t-1}(\boldsymbol{\theta})).$

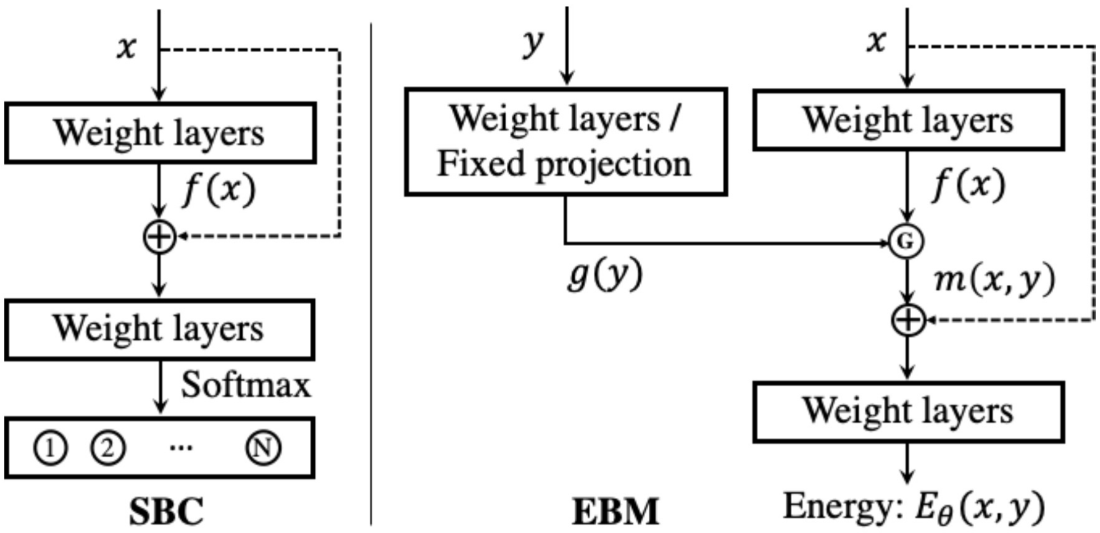
  $q_t(\boldsymbol{\theta}) = \prod_{d=1}^{D} \mathcal{N}(\theta_{t,d}; \mu_{t,d}, \sigma^2_{t,d}).$

- Compare to EWC: Maintains uncertainty throughout training.

Nguyen et al. Variational Continual Learning. ICLR 2018.

EBM for Continual Learning

- Softmax layer is known to be sensitive to distribution shift.



Li et al. Energy-Based Models for Continual Learning. CoLLAs 2022.

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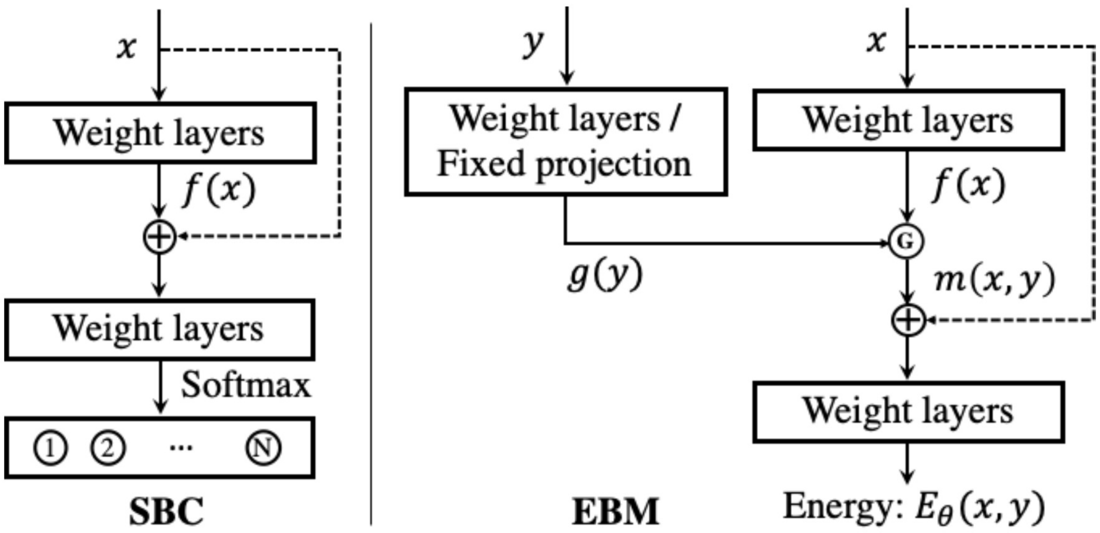

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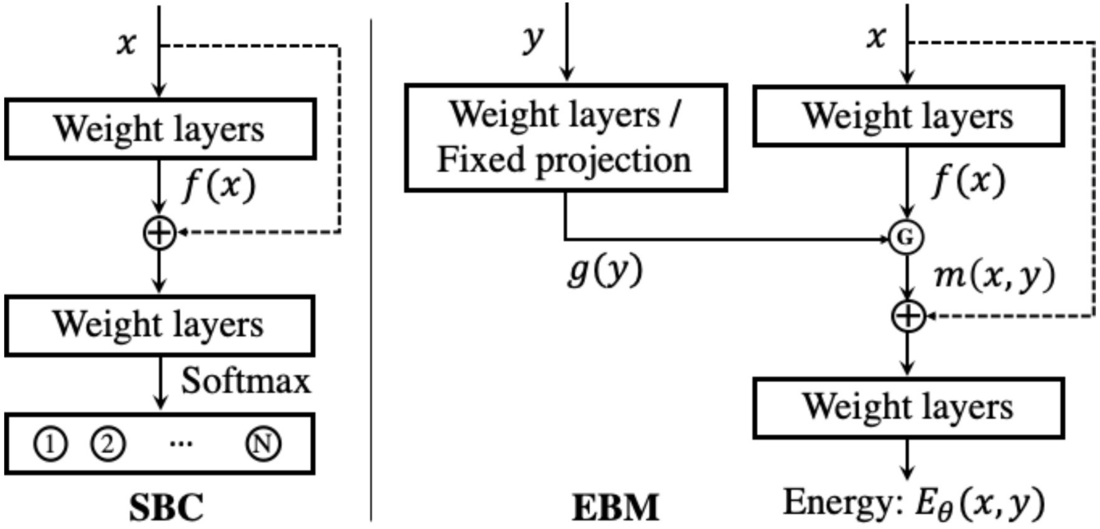

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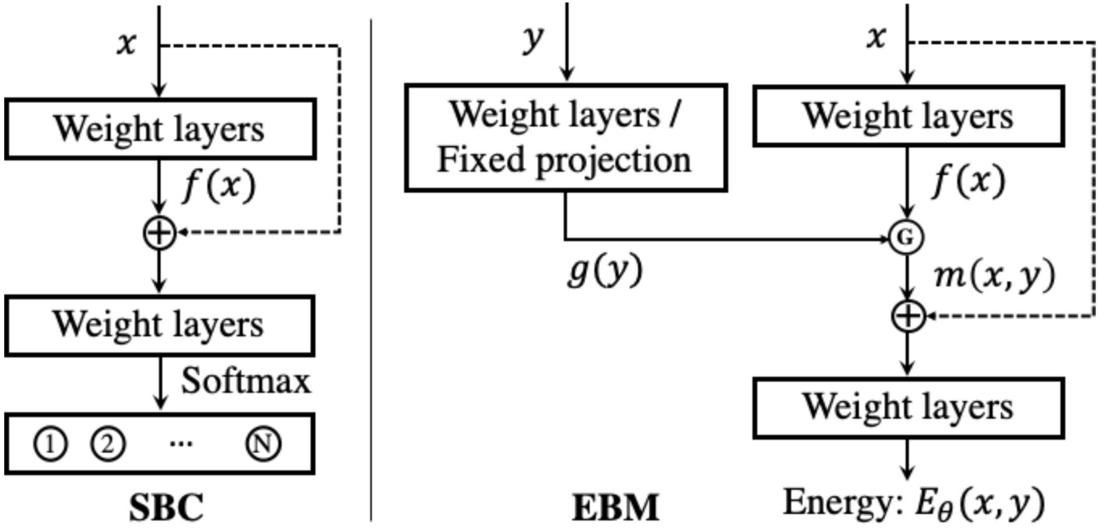
EBM for Continual Learning

- Softmax layer is known to be sensitive to distribution shift.
- A common approach is to use nearest mean classifier.
- Can be generalized to EBMs
- Energy between inputs and labels:

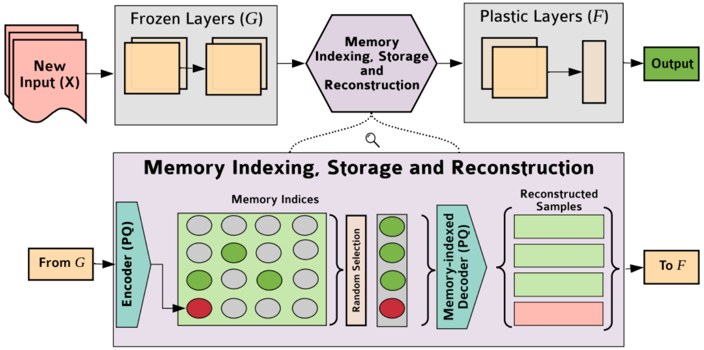
$$m(\mathbf{x}, y) = G(f(\mathbf{x}), g(y))$$



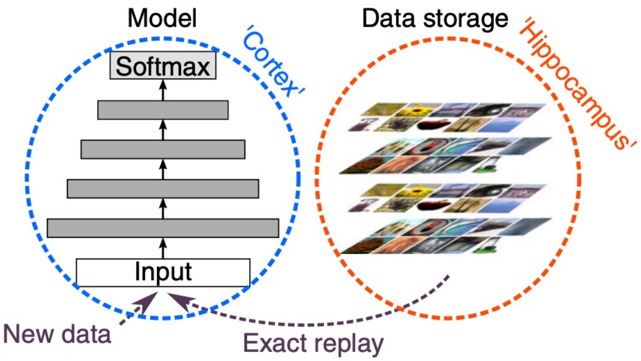
$$L_{CD}(\theta) = \mathbb{E}_{(\mathbf{x},y,y^-)\sim p_D}[E_\theta(\mathbf{x}, y) - E_\theta(\mathbf{x}, y^-)].$$

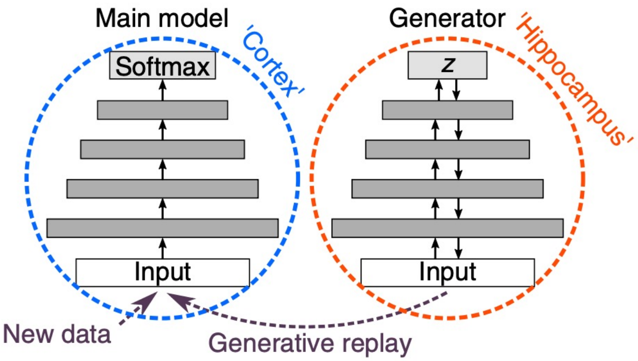
Li et al. Energy-Based Models for Continual Learning. CoLLAs 2022.

Replay

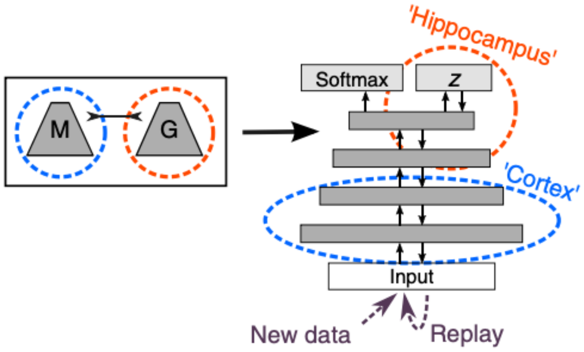


- Store raw data, representations, or train a generative model





Feedback Connections



Rebuffi et al. iCaRL: Incremental Classifier and Representation Learning. CVPR 2017.
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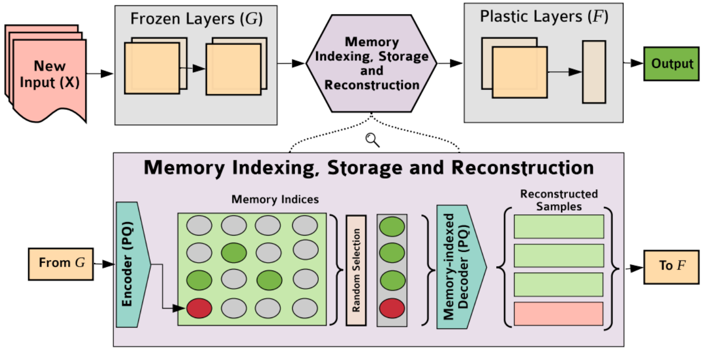
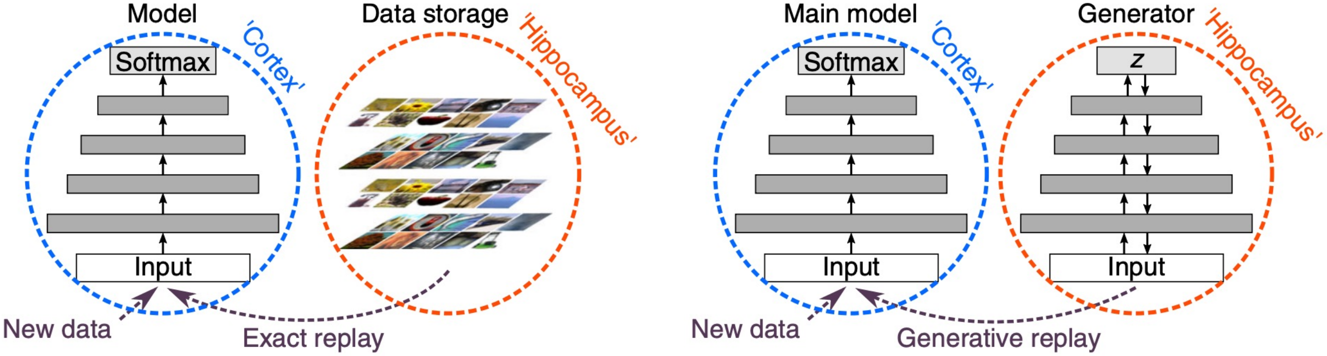


Replay

- Store raw data, representations, or train a generative model
- Coreset selection $\mu \leftarrow \frac{1}{n} \sum_{x \in X} \varphi(x)$ $\quad p_k \leftarrow \operatorname*{argmin}_{x \in X} \left\| \mu - \frac{1}{k}[\varphi(x) + \sum_{j=1}^{k-1} \varphi(p_j)] \right\|.$



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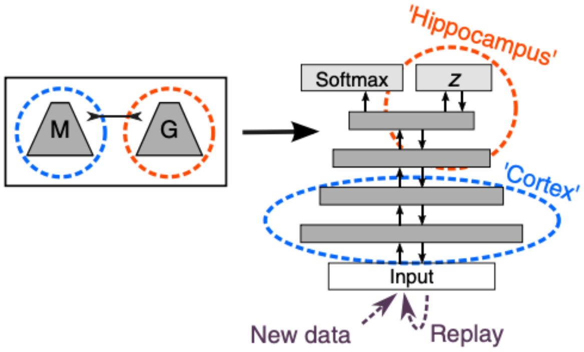

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Knowledge Distillation

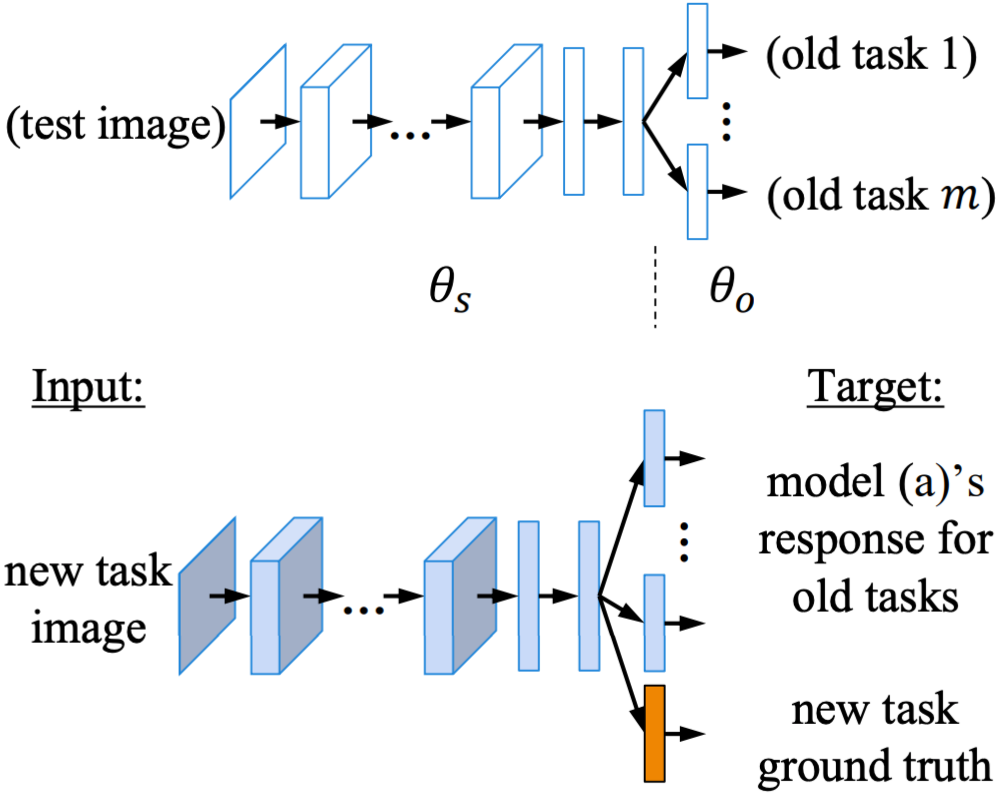
- Instead of saving the data points, we can also save the previous model checkpoint.

$$y_o = f(x_n; \theta_o).$$

$$\hat{y}_o = f(x_n; \theta_n).$$

$$\mathcal{L}(y_o, \hat{y}_o)$$

- Use new data points and old weights to "distill"



Li & Hoiem. Learning without Forgetting. ECCV 2016.

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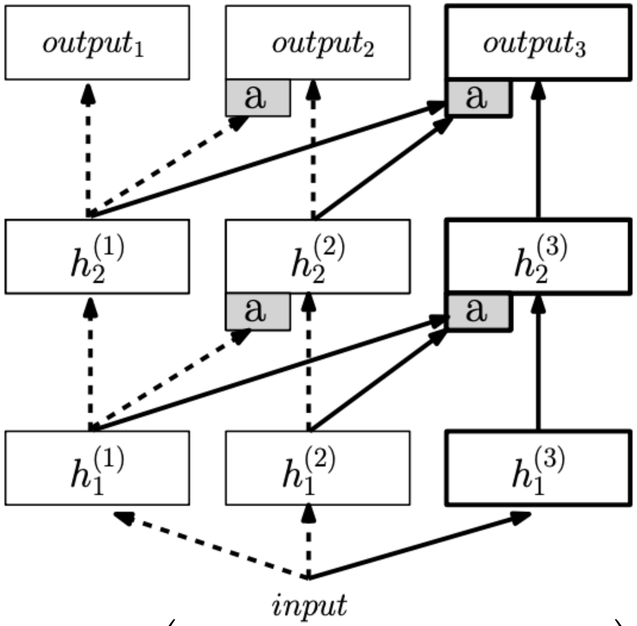
Architecture Expansion



$$h_i^{(k)} = f\left(W_i^{(k)} h_{i-1}^{(k)} + \sum_{j<k} U_i^{(k:j)} h_{i-1}^{(j)}\right).$$

Rusu et al. Progressive Neural Networks. NIPS 2016 Deep Learning Symposium.
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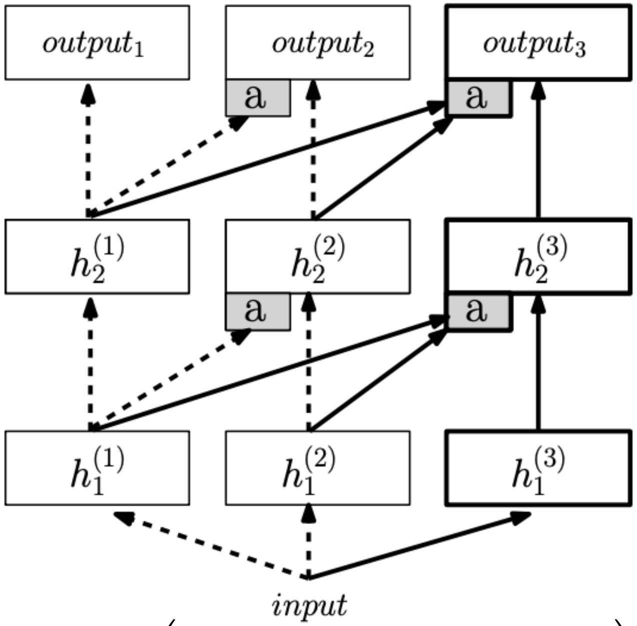

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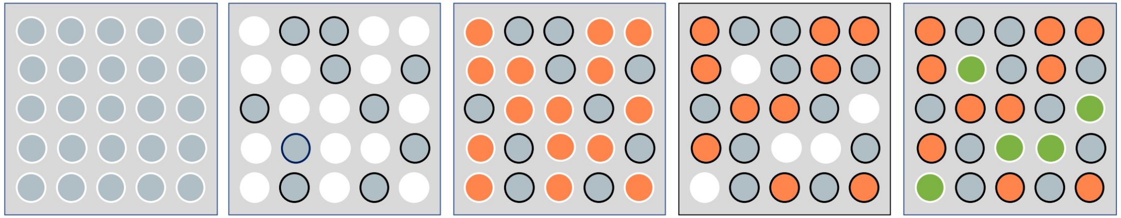

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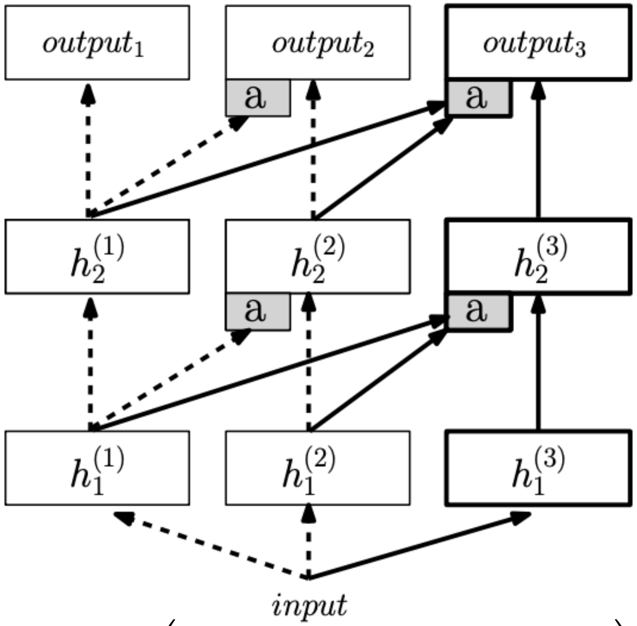

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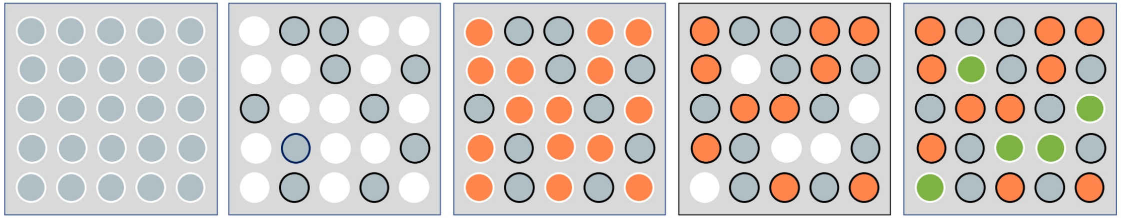

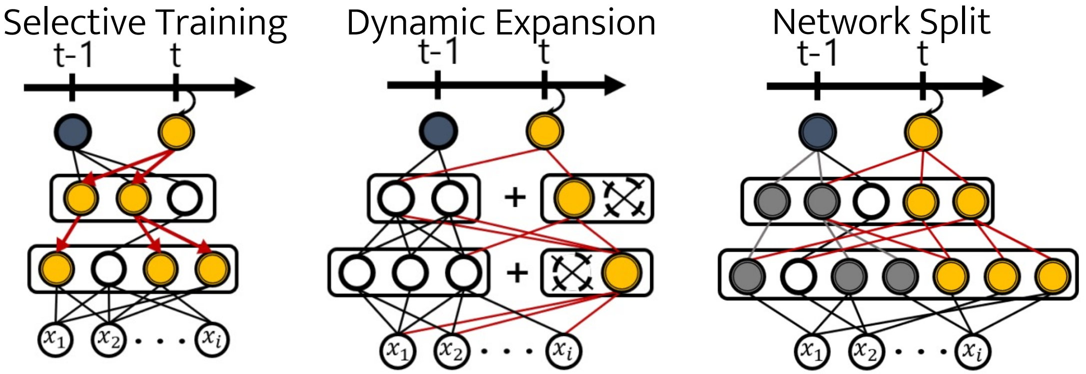

Selective Training

$$\mathcal{L}(\boldsymbol{W}_L^t, \boldsymbol{W}_{1:L-1}^{t-1}, \mathcal{D}_t) + \mu\|\boldsymbol{W}_L^t\|_1$$

Dynamic Expansion | Group sparsity

$$\mathcal{L}(\boldsymbol{W}_l^{\mathcal{N}}, \boldsymbol{W}_l^{t-1}, \mathcal{D}_t) + \mu\|\boldsymbol{W}_l^{\mathcal{N}}\|_1 + \gamma\sum_g\|\boldsymbol{W}_{l,g}^{\mathcal{N}}\|_2$$

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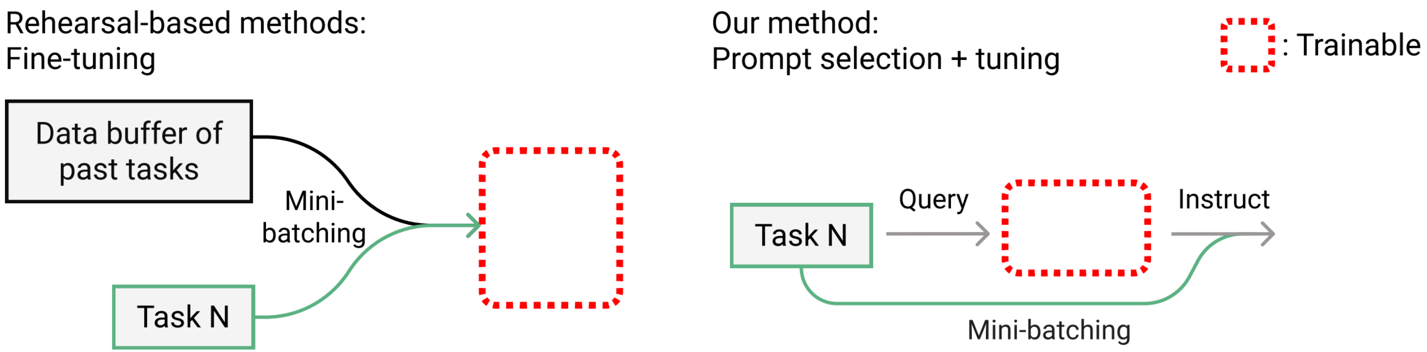
Adapting Pretrained Models

- Pretrained models have general knowledge that can be adapted to a continual stream of tasks.



Wang et al. Learning to Prompt for Continual Learning. CVPR 2022.

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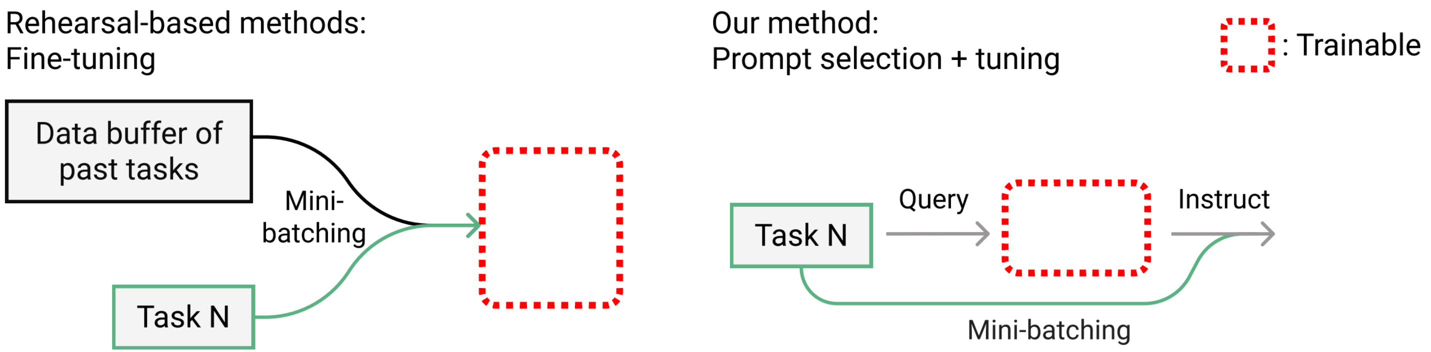

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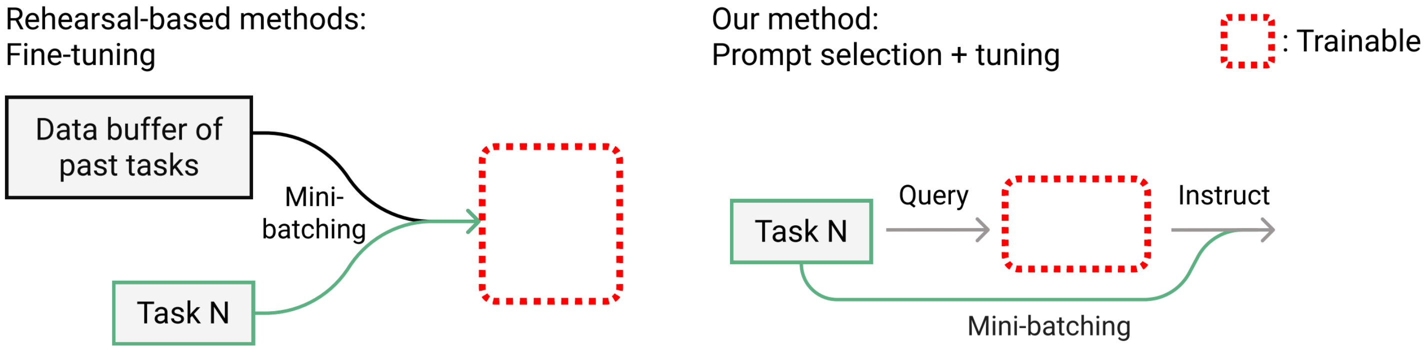
Adapting Pretrained Models

- Pretrained models have general knowledge that can be adapted to a continual stream of tasks.
- Learn adaptation parameters for each task and store these as "task embeddings."
- Main model is frozen.



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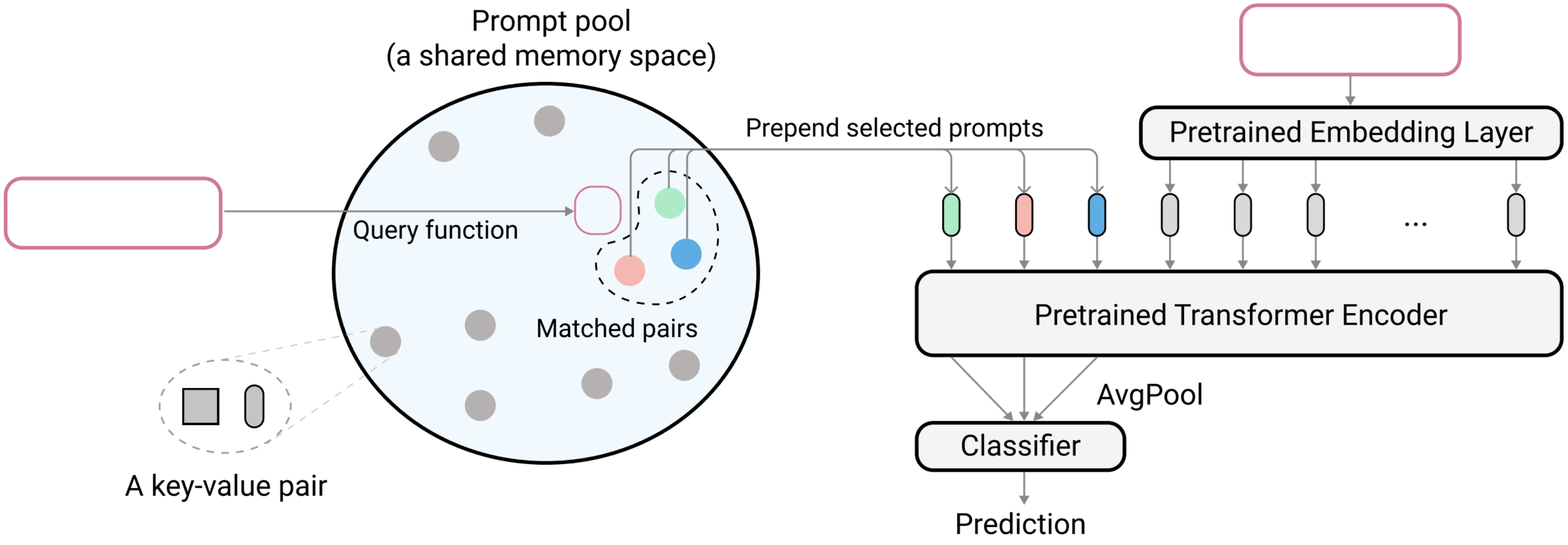
Learning to Prompt

Prompt pool (slot memory)

$$\{(k_1, p_1), \ldots, (k_M, p_M)\}$$



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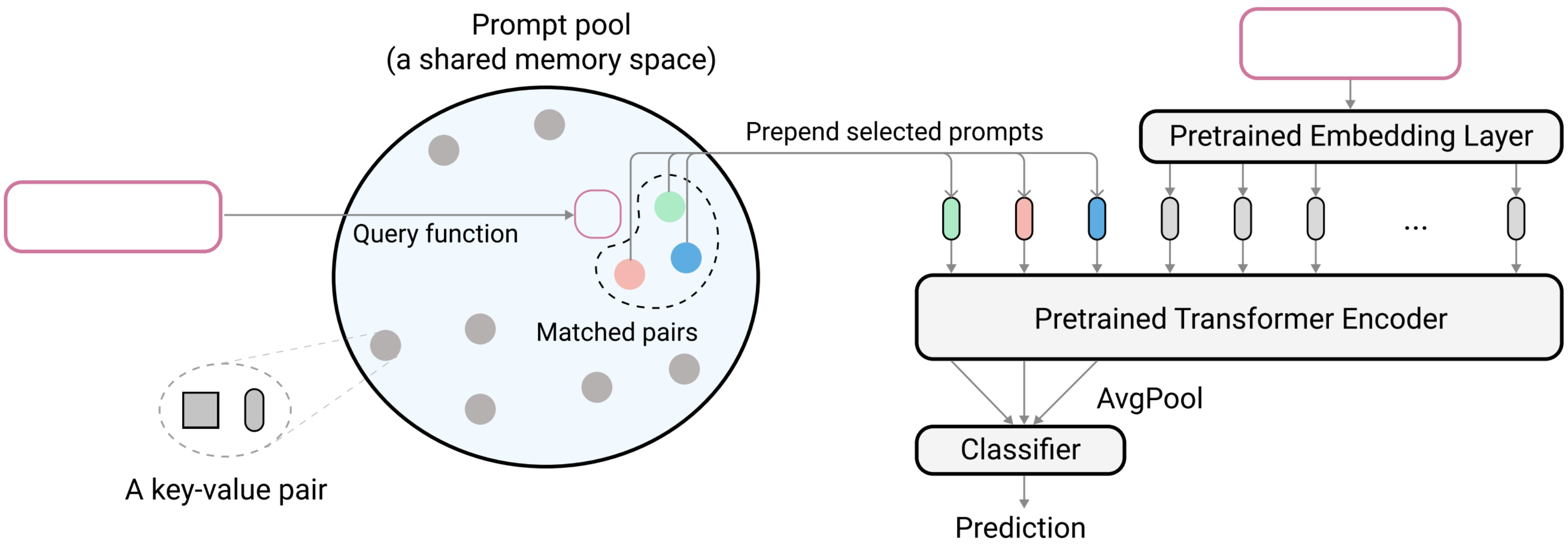

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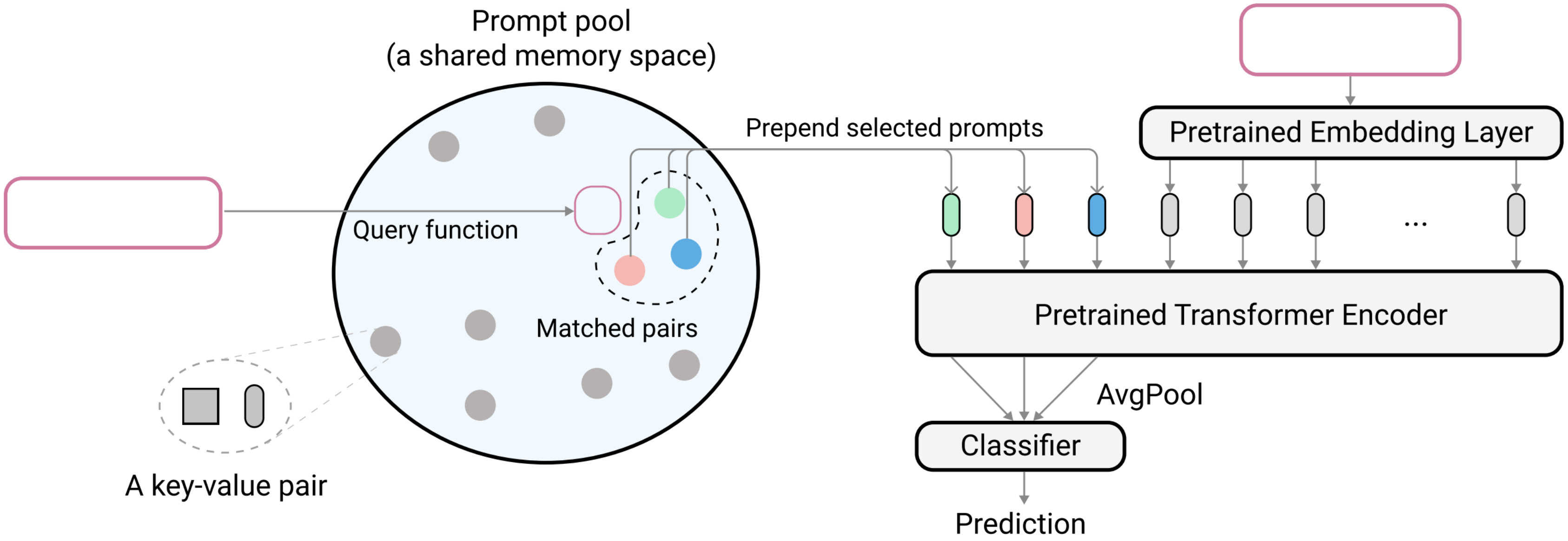
$$\min_{P,K,\phi} \mathcal{L}(g_\phi(x), y) + \lambda \sum_{i \in K_s} \gamma(q(x), k_i)$$



Wang et al. Learning to Prompt for Continual Learning. CVPR 2022.

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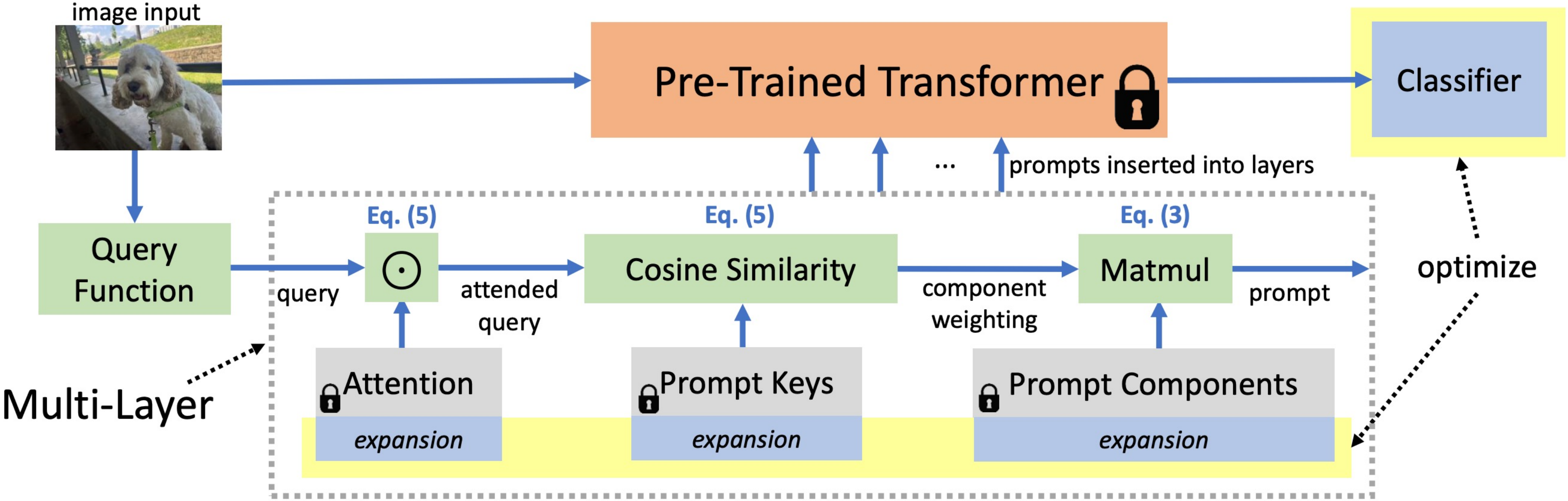
Learned Prompt Query

- The query function can be end-to-end learned.

$$\alpha = \{\gamma(q(x) \odot A_1, K_1), \ldots, \gamma(q(x) \odot A_M, K_M)\}$$



Smith et al. CODA-Prompt: COntinual Decomposed Attention-based Prompting for Rehearsal-Free Continual Learning. CVPR 2023.

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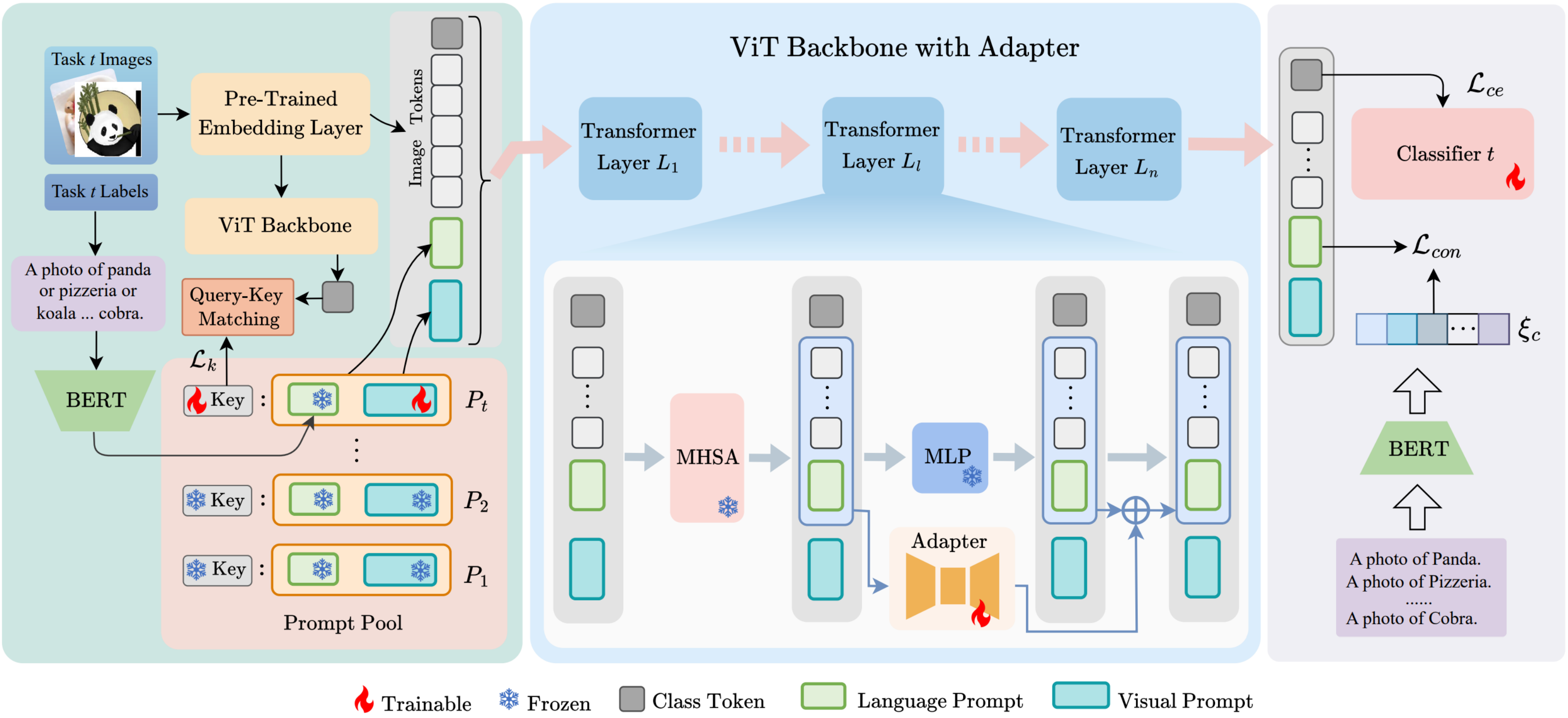
Multimodal Semantic Prompts



Yin et al. Adapter-Enhanced Semantic Prompting for Continual Learning. arXiv 2024.

Continual Learning and Memory

- Fragility of feedforward gradient descent of the entire networks

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- But what if representations also need to be built sequentially?
- It’s also plausible that representations are just “deeper memory.”

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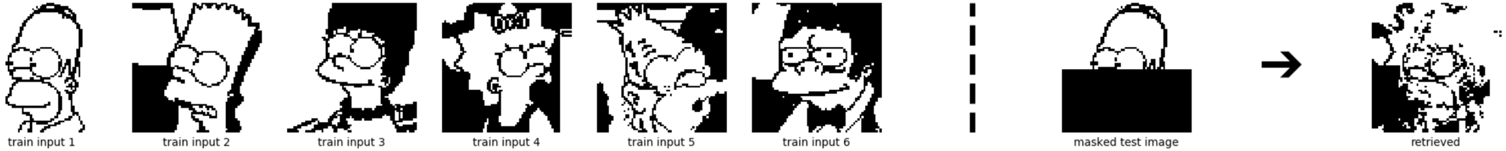
Associative Memory

- Memory aims to store content for easy retrieval



Krotov & Hopfield. Dense Associative Memory for Pattern Recognition. NIPS 2016.
https://ml-jku.github.io/hopfield-layers/

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Associative Memory

- Memory aims to store content for easy retrieval
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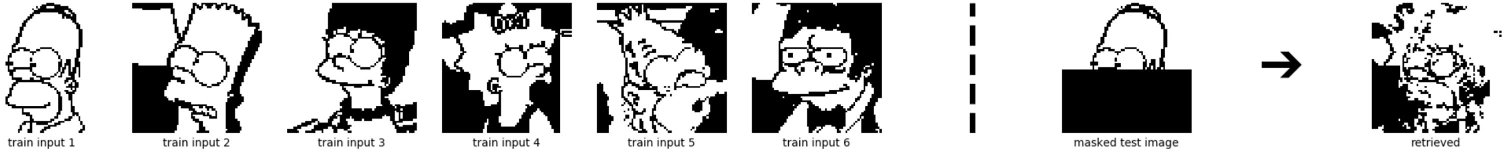

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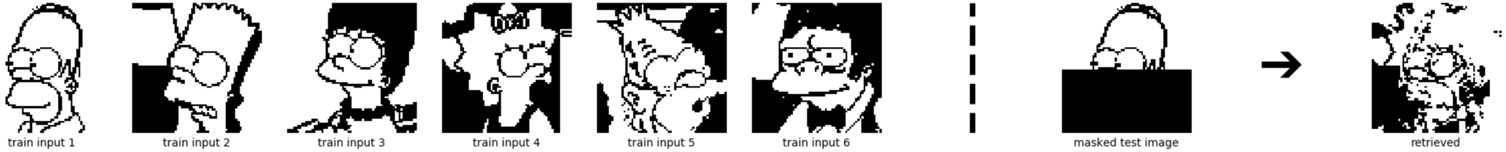
$$E = -\frac{1}{2} \sum_{i,j=1}^{N} s_i W_{ij} s_j , \qquad W_{ij} = \sum_{k=1}^{K} \xi_i^k \xi_j^k .$$

"Superposition" of k slots
Hebbian learning



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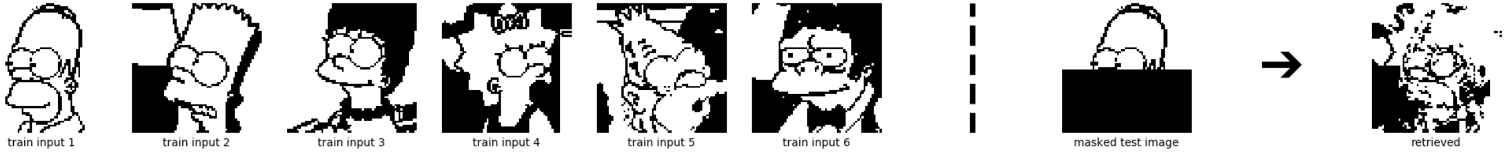

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”Superposition” of k slots
Hebbian learning

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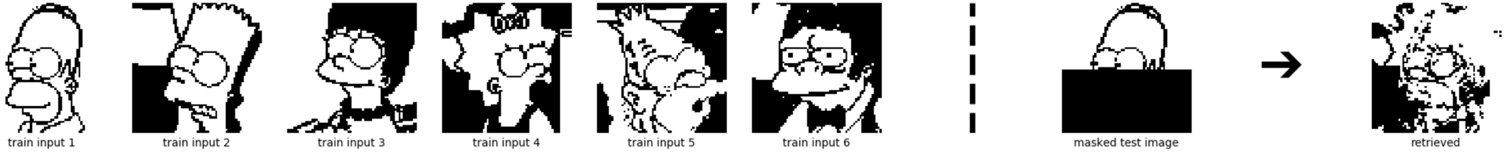
  - Retrieval: $s_i = \mathrm{sign}(\sum_j W_{ij} s_j)$ Storage: $C \approx \frac{d}{2\log(d)} = 0.14d.$



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Memory Duality

- Duality with a feedforward network.

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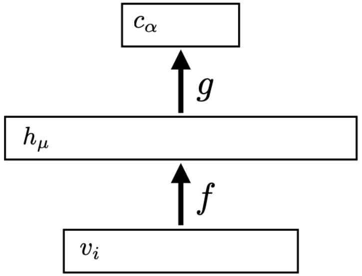

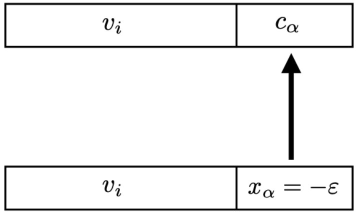

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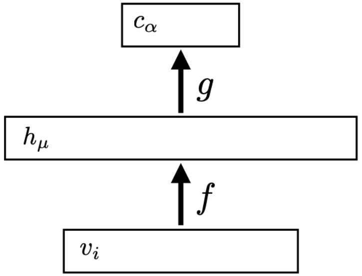

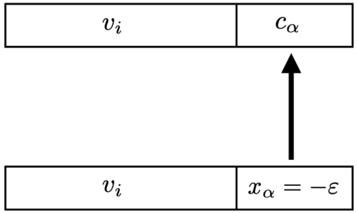

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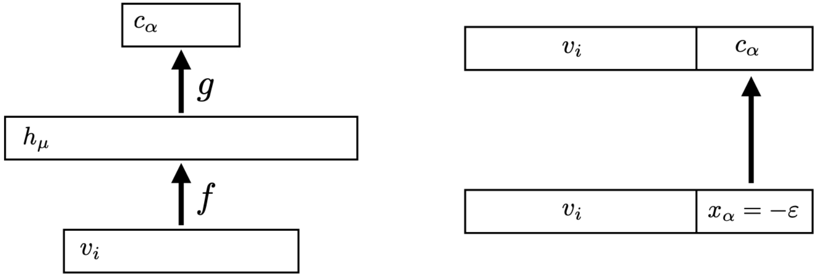
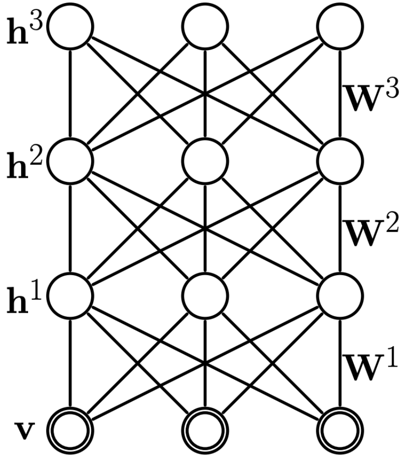


Deep Boltzmann Machine



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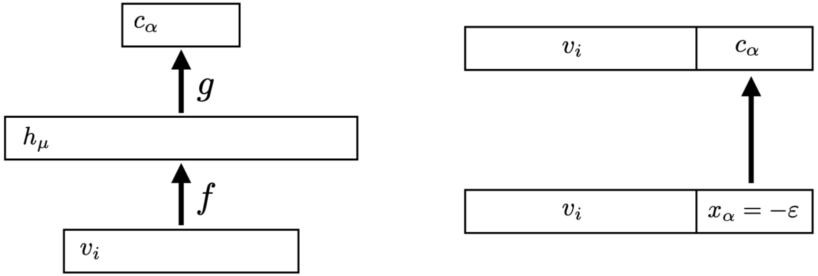

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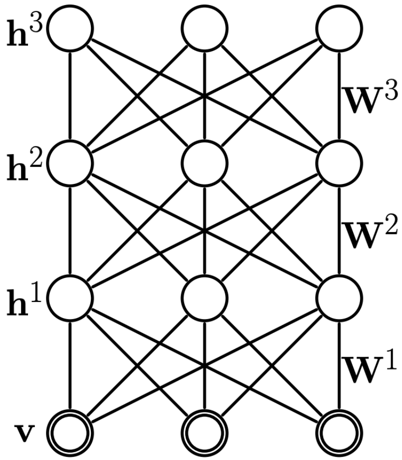

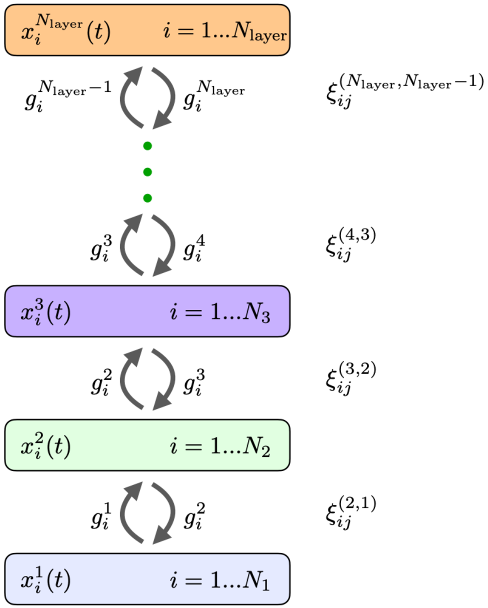

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Relation to Transformers

- General form:

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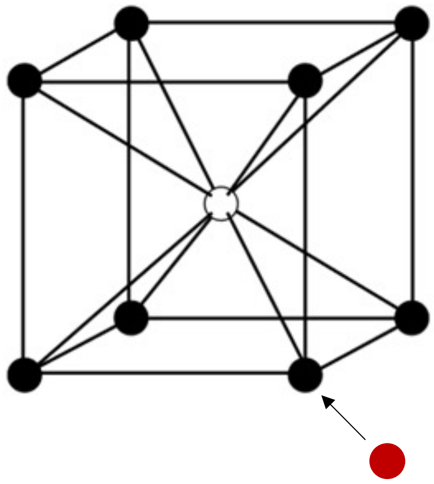

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- When $F(z) = z^2$ it gives the classic HN.

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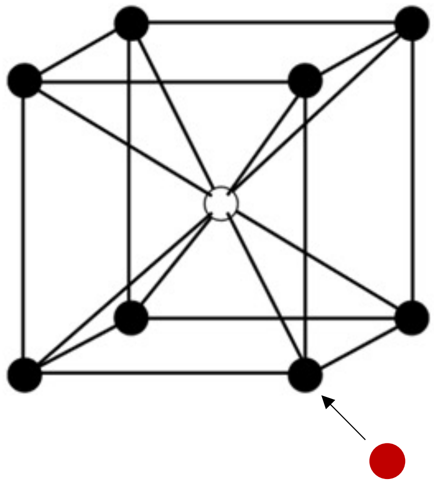
Ramsauer et al. Hopfield Networks is All You Need. ICLR 2021.

NYU

Relation to Transformers



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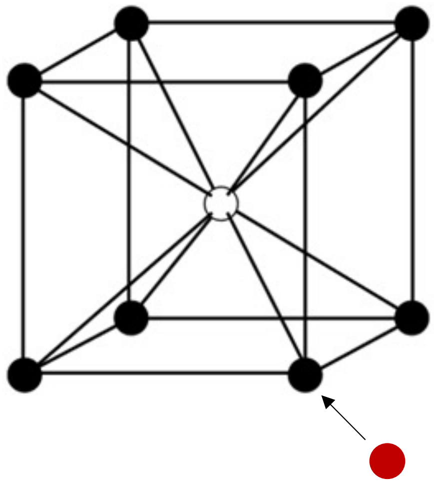

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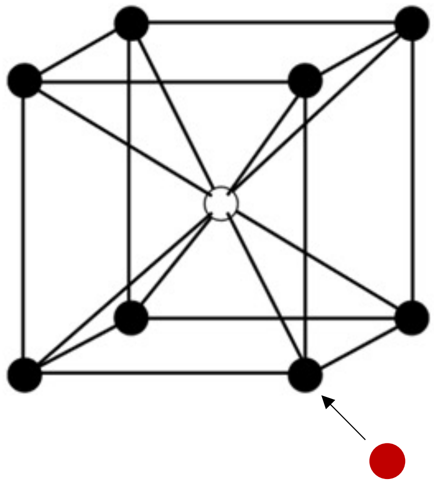

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$$E = -\mathrm{logsumexp}(\beta, \boldsymbol{\Xi}^\top \boldsymbol{s}) + \frac{1}{2} \boldsymbol{s}^\top \boldsymbol{s} + \beta^{-1} \log N + \frac{1}{2} M^2.$$

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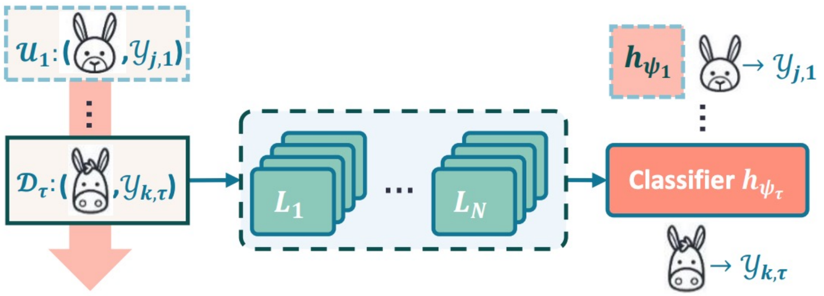
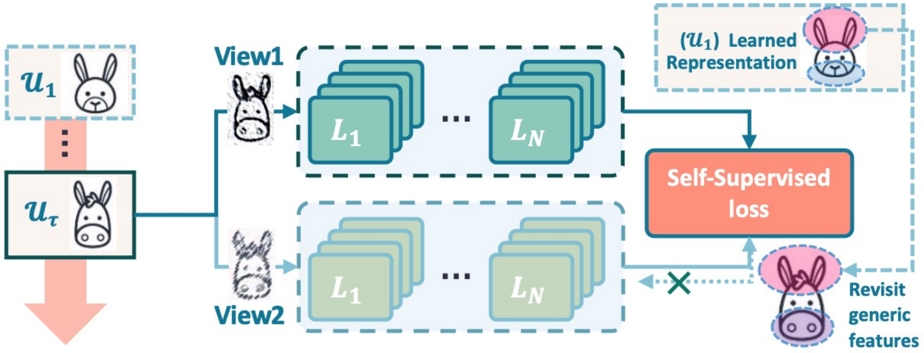
Continual Self-Supervised Learning

- Learning from a stream of unlabeled inputs.



SUPERVISED CONTINUAL LEARNING (SCL)



UNSUPERVISED CONTINUAL LEARNING (UCL)

Madaan et al. Representational Continuity for Unsupervised Continual Learning. ICLR 2022.

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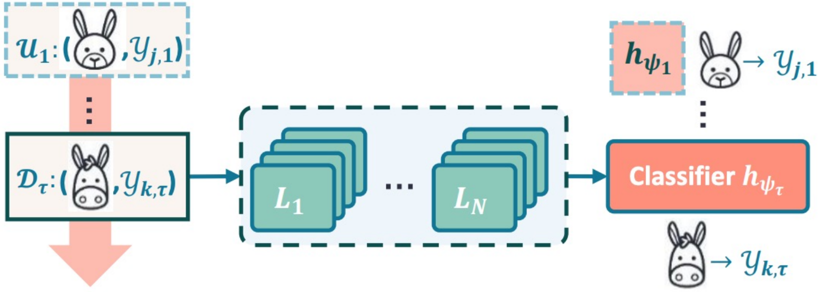
Continual Self-Supervised Learning

- Learning from a stream of unlabeled inputs.
- Bring SSL to the dynamic world.



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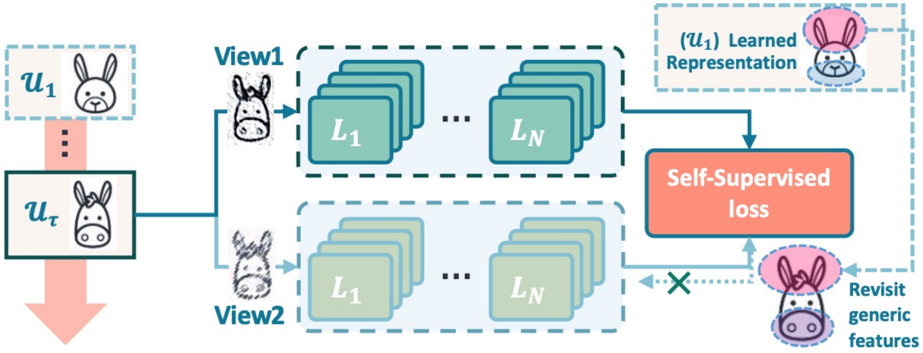

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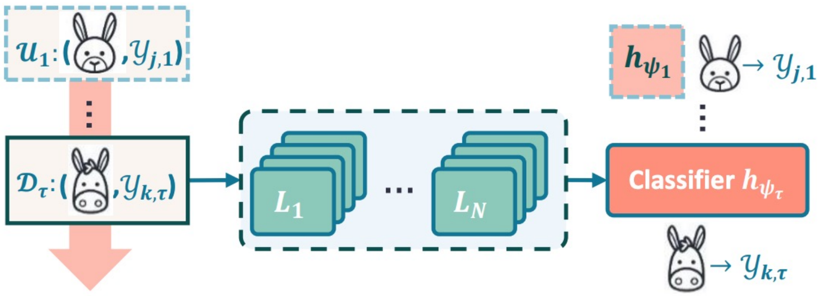
Continual Self-Supervised Learning

- Learning from a stream of unlabeled inputs.
- Bring SSL to the dynamic world.
- SSL can still suffer from distributional shifts.



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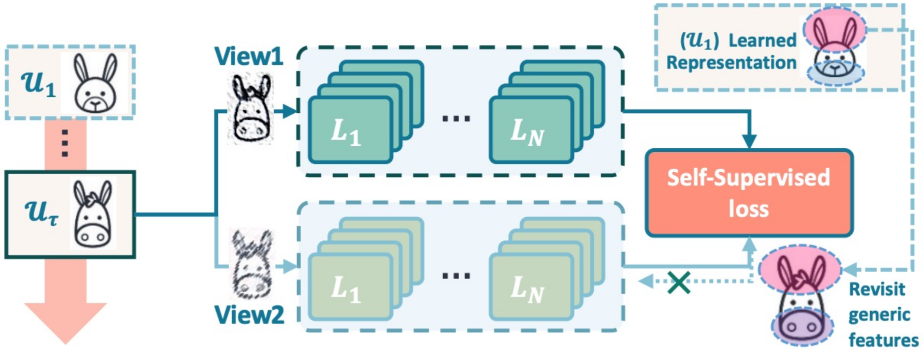

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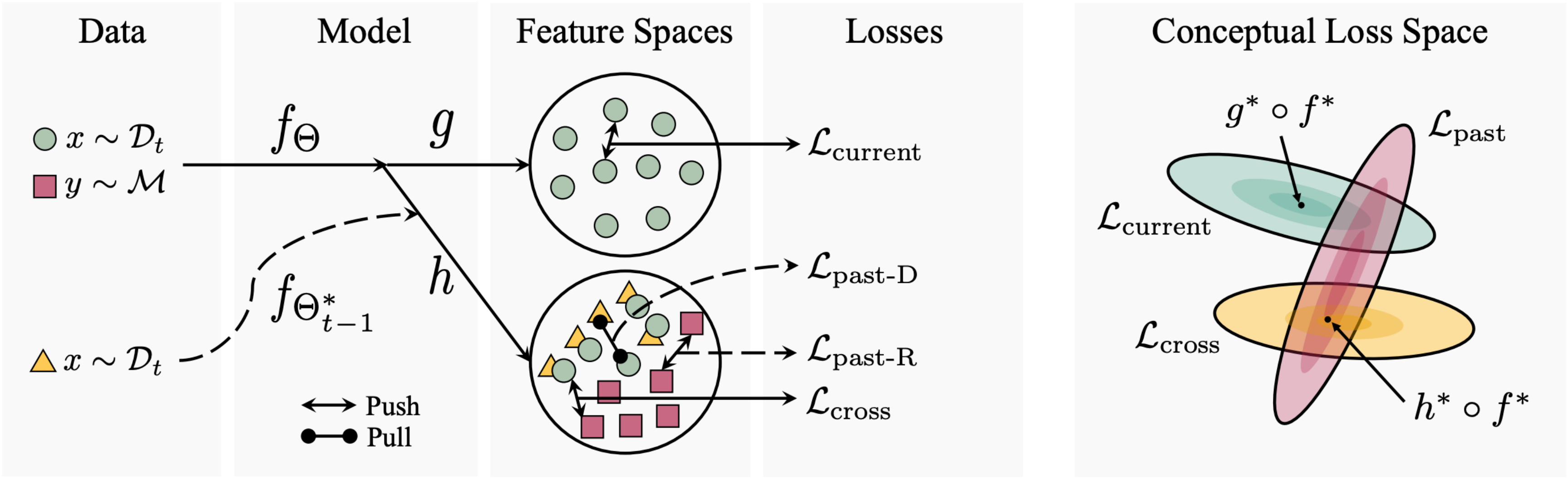
NYU

Continual Self-Supervised Learning

- Integrating learning objectives of both past and present.

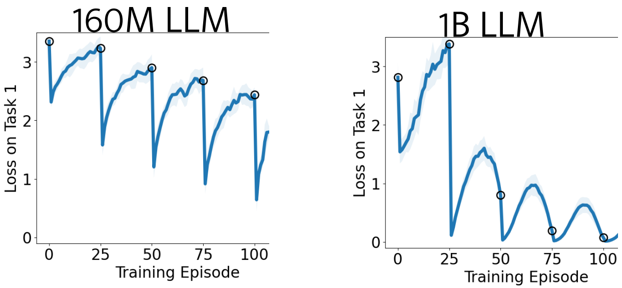
$$\underbrace{\mathcal{L}(X; g \circ f_t)}_{\text{Current}} + \underbrace{\mathcal{L}(X, X; h \circ f_{t-1}, h \circ f_t)}_{\text{Past (Distillation)}} + \underbrace{\mathcal{L}(Y; h \circ f_t)}_{\text{Past (Replay)}} + \underbrace{\mathcal{L}(X, Y; h \circ f_t)}_{\text{Cross-Consolidation}}$$



Zhang et al. Integrating Present and Past in Unsupervised Continual Learning. CoLLAs 2024.

Outlooks

- Understand continual learning at scale

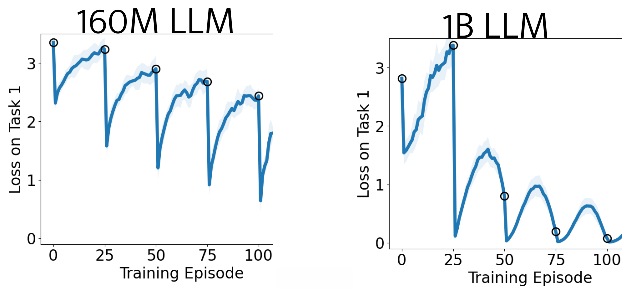


[Mayo et al. 2023]

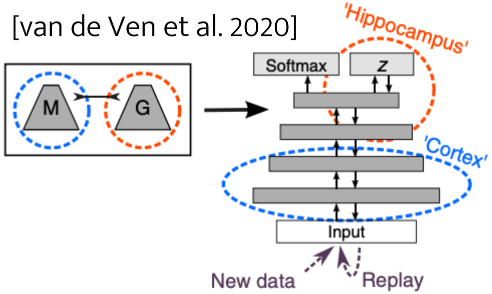
Outlooks

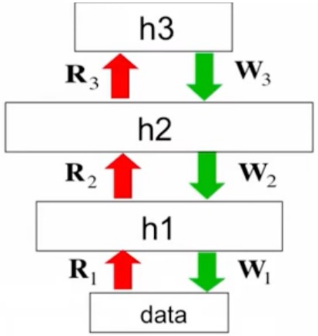
- Understand continual learning at scale

- Unified learning architecture, objective and replay, role of sleep



[van de Ven et al. 2020]



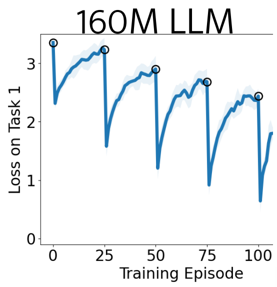


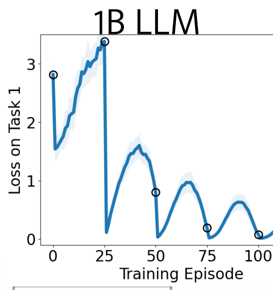
[Hinton et al. 1995]

[Mayo et al. 2023]

Outlooks

- Understand continual learning at scale
- Unified learning architecture, objective and replay, role of sleep
- Continual learning with real world structure





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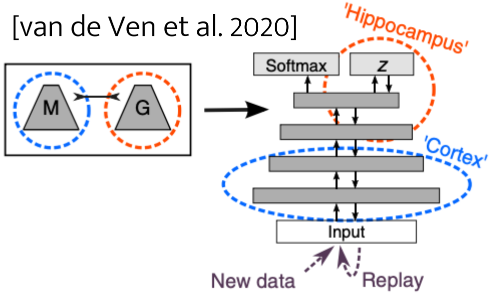
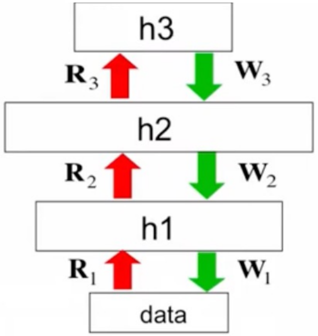
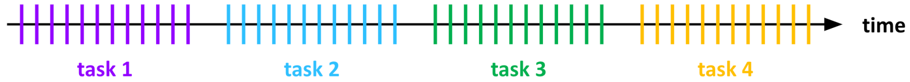




[Hinton et al. 1995]

Typical setting in continual, few-shot, transfer, and meta-learning



Typical setting of human learning [Mayo et al. 2023]

Tasks are interspersed and recur
No opportunity to master one before confronted with another

Summary: Continual Learning

- Regularization, Distillation, Architecture Expansion/Isolation

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